# Spectral Theory of Self-adjoint Higher Order Differential Operators With Eigenvalue Parameter Dependent Boundary Conditions

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A thesis submitted to the Faculty of Science, University of the Witwatersrand, in fulfillment of the requirements for the degree of PhD.

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# Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree of PhD in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

Bertin Zinsou

This day of 23 April 2012, in Johannesburg, South Africa.

# Abstract

We consider on the interval $[0, a]$, firstly fourth-order differential operators with eigenvalue parameter dependent boundary conditions and secondly a sixth-order differential operator with eigenvalue parameter dependent boundary conditions. We associate to each of these problems a quadratic operator pencil with self-adjoint operators. We investigate the spectral proprieties of these problems, the location of the eigenvalues and we explicitly derive the first four terms of the eigenvalue asymptotics.

# Praisings

I will praise you, O Lord, with all my heart; I will tell all your wonder.

I will be glad and rejoice in you; I will sing praise to your name, O Most High.

Ps 9: 1–2.

The Lord is my strength and my shield; my heart trusts in him, and I am helped.

My heart leaps for joy and I will give thanks to him in song.

Ps 28: 7.

My heart is steadfast, O God; I will sing and make music with all my soul.

Awake, harp and lyre! I will awake the dawn.

I will praise you, O Lord, among the nations; I will sing of you among the peoples.

For great is your love, higher than the heavens; your faithfulness reaches the skies.

Ps 108: 1–4.

*To my late mother Delphine Ouensavi Zinsou , my sons Senan and Sègnon and my wife Hermance Kokode Zinsou*

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# Chapter 1

# Introduction

Differential equations were introduced in the seventeenth century to describe fundamental laws in Physics [15]. Currently differential equations are widely used in Physics as well as in several areas such as Engineering, Geophysics, Geography, Economics, to mention but a few. Differential equations are applied in hydrodynamics to predict fluid migration. Furthermore, they are used in electricity and mechanics. Differential equations have applications, which are used on optimal design of space shuttles, aircrafts and ships. Differential equations are used in climatology for studying climatic changes. Furthermore, differential equations are used to forecast population and economic trends.

The study of differential equations involves applications of differential operators. It is to be noted that that the scope of differential operators is very broad. This scope covers several topics such as eigenvalues, eigenfunctions or eigenvectors, boundary conditions, regularity, asymptotics, boundedness, normality, compactness, minimality, maximality, closed and closable operators, symmetric operators, adjoint operators, self-adjoint operators and operator graphs. Also the scope of differential operators involves notions such as resolvent set and spectra. Spectra can be subdivided into three disjoint sets: point spectrum or discrete spectrum, continuous spectrum and residual spectrum.

The Sturm-Liouville operators are well investigated and developed, while higher order differential operators, in particular with eigenvalue parameter dependent boundary conditions, are much less investigated and understood. Singular and (quasi-)regular problems, as in the Sturm-Liouville case, are considered. They are distinguished by their spectral properties. Discussions on fourth-order differential operators can be found in [6, 7]. The minimal differential operator of $n$th order self-adjoint differential operators must be symmetric, see [37, 51] for necessary and sufficient conditions. General characterizations of self-adjoint boundary conditions have been given in [47, 48]. Different characteristics of higher order differential operators whose boundary conditions depend on the eigenvalue parameter, including spectral asymptotics and basic properties, have been studied in [8, 9, 22, 23, 30, 46].

The generalized Regge problem is realized by a second order differential operator which depends quadratically on the eigenvalue parameter and which has eigenvalue parameter dependent boundary conditions, see [43]. The particular feature in this case is that the coefficient operators of this pencil are self-adjoint, and it is shown that this gives some a priori knowledge about the location of the spectrum. In [34] this approach has been extended to a fourth order differential equation describing small transversal vibrations of a homogeneous beam compressed or stretched by a force $g$. Separation of variables leads to an ordinary fourth order differential equation with eigenvalue parameter dependent boundary conditions, where the differential equation depends quadratically on the eigenvalue parameter. This problem is described by the differential equation:

$$y^{(4)} - (gy')' = \lambda^2 y \tag{1.1}$$

and one particular set of boundary conditions which are

$$y(\lambda, 0) = 0, \tag{1.2}$$

$$y''(\lambda, 0) = 0, \tag{1.3}$$

$$y(\lambda, a) = 0, \tag{1.4}$$

$$y''(\lambda, a) + i\alpha\lambda y'(\lambda, a) = 0, \tag{1.5}$$

where $a > 0$, $\alpha \geq 0$ and $g \in C^1[0, a]$ is a real value function. In the Hilbert space $L_2(0, a) \oplus \mathbb{C}$, this problem is represented by the quadratic operator pencil

$$L(\lambda) = \lambda^2 M - i\alpha\lambda K - A \tag{1.6}$$

where

$$K = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \text{ and } \mathrm{M} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

are bounded self-adjoint operators with domain

$$\mathscr{D}(K) = \mathscr{D}(M) = L_2(0, a) \oplus \mathbb{C},$$

while $A$ is an unbounded operator defined by

$$A\widetilde{y} = \begin{pmatrix} y^{(4)} - (gy')' \\ y''(a) \end{pmatrix}, \quad \widetilde{y} = \begin{pmatrix} y \\ y'(a) \end{pmatrix}$$

with

$$\mathscr{D}(A) = \left\{ \widetilde{y} = \begin{pmatrix} y \\ y'(a) \end{pmatrix}, y \in W_4^2(0, a), y(0) = y''(0) = y(a) = 0 \right\},$$

see Definition 2.36 for the definition of $W_4^2(0, a)$.

Sixth-order boundary value problems arise in astrophysics, the narrow convection layers bounded by stable layers believed to surround A-type stars may be modeled by sixth-order boundary value problems [28]. When an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in, when this instability is as ordinary convection, the differential equation is an ordinary sixth-order differential equation [10]. The theorems stating conditions of the existence and uniqueness of solutions of sixth-order boundary problems can be found in [1]. Numerical methods of sixth-order boundary value problems are widely investigated. Numerical methods and other techniques of the investigation of sixth-order boundary value problems can be found in [2, 3, 4, 5, 18, 19, 28, 32, 33]. However the literature does not reveal any study about the spectral theory of sixth-order differential operators. We may assume that there is no study on spectral theory of sixth-order differential operators or only a few of them exist.

Since higher order differential operators, in particular with eigenvalue parameter dependent boundary conditions, are much less investigated and understood, it becomes necessary to conduct studies on higher order differential operators with eigenvalue parameter dependent boundary conditions, namely self-adjoint fourth-order differential operators with eigenvalue parameter dependent boundary conditions and self-adjoint sixth-order differential operator with eigenvalue parameter dependent boundary conditions.

We have considered in [39] on the interval $[0, a]$ the eigenvalue problem

$$y^{(4)} - (gy')' = \lambda^2 y, \tag{1.7}$$

$$B_j(\lambda)y = 0, \ j = 1, 2, 3, 4, \tag{1.8}$$

where $a > 0$, $g \in C^1[0, a]$ is a real value function and (1.8) are separated boundary conditions where the $B_j(\lambda)$ are constant or dependent on $\lambda$ linearly. We recall that the quasi-derivatives associated with (1.7) are given by

$$y^{[0]} = y, \ y^{[1]} = y', \ y^{[2]} = y'', \ y^{[3]} = y^{(3)} - gy', \ y^{[4]} = y^{(4)} - (gy')',$$

see Definition 2.3. The boundary conditions (1.8) are taken at the endpoint 0 for $j = 1, 2$ and at the endpoint $a$ for $j = 3, 4$. We have assumed for simplicity that either $B_j(\lambda)y = y^{[p_j]}(a_j) + i\varepsilon_j\alpha\lambda y^{[q_j]}(a_j)$ or $B_j(\lambda)y = y^{[p_j]}(a_j)$, where $a_j = 0$ for $j = 1, 2$ and $a_j = a$ for $j = 3, 4$, $\alpha > 0$, $0 \le p_j, q_j \le 3$ and $\varepsilon_j \in \{-1, 1\}$. We have defined

$$\Theta_1 = \{s \in \{1, 2, 3, 4\} : B_s(\lambda) \text{ depends on } \ \lambda\}, \ \Theta_0 = \{1, 2, 3, 4\} \setminus \Theta_1,$$

$$\Theta_1^0 = \Theta_1 \cap \{1, 2\}, \quad \Theta_1^a = \Theta_1 \cap \{3, 4\}$$

and put

$$k = |\Theta_1|. \tag{1.9}$$

We have denoted the collection of the four boundary conditions (1.8) by $U$ and have defined the following operators related to $U$:

$$U_0 y = (y^{[p_j]}(a_j))_{j \in \Theta_1} \ \text{ and } \ U_1 y = (\varepsilon_j y^{[q_j]}(a_j))_{j \in \Theta_1}.$$

We have associated the quadratic operator pencil

$$L(\lambda) = \lambda^2 M - i\alpha\lambda K - A(U) \tag{1.10}$$

in the space $L_2(0, a) \oplus \mathbb{C}^k$ with the problem, where

$$K = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \text{ and } M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

are bounded self-adjoint operators. The operator $A(U)$, which will be given below, depends on the boundary conditions $U$. The operators $K$ and $M$ also depend on $U$ since the number $k$ in $L_2(0,a)\oplus\mathbb{C}^k$ is the number of $\lambda$-dependent boundary conditions. The maximal differential operator $A_{\max}$ associated with the boundary value problem (1.7) and the boundary conditions (1.8) is defined on $L_2(0,a)\oplus\mathbb{C}^k$ by

$$\mathscr{D}(A_{\max})=\left\{\widetilde{y}=\begin{pmatrix} y \\ U_1y\end{pmatrix} : y\in W_4^2(0,a)\right\},\quad A_{\max}\widetilde{y}=\begin{pmatrix} y^{[4]} \\ U_0y\end{pmatrix},\quad \widetilde{y}\in\mathscr{D}(A_{\max}).$$

The operator $A(U)$ is defined in $L_2(0,a)\oplus C^k$ by

$$\mathscr{D}(A(U))=\left\{\widetilde{y}=\begin{pmatrix} y \\ U_1y\end{pmatrix} : y\in W_4^2(0,a), y^{[p_j]}(a_j)=0 \text{ for } j\in\Theta_0\right\},\tag{1.11}$$

$$(A(U))\widetilde{y}=\begin{pmatrix} y^{[4]} \\ U_0y\end{pmatrix}\quad \text{for } \widetilde{y}\in\mathscr{D}(A(U)).\tag{1.12}$$

See Definition 2.36 for the definition of $W_4^2(0,a)$. The definition of the operator $A(U)$ shows that $A(U)$ is the restriction of $A_{\max}$ with respect to those boundary conditions in $U$ which do not depend on $\lambda$. We have investigated in [39] a class of boundary conditions for which necessary and sufficient conditions were obtained such that the associated operator pencil consists of self-adjoint operators. We continue during this study the work of [39] in the direction of [34] to derive eigenvalue asymptotics associated with boundary conditions which lead to self-adjoint operator representations. In this case, we consider the eigenvalue problems (1.7)–(1.8) where the boundary conditions are $B_1y=y^{[p_1]}(0)=0$ and $B_2y=y^{[p_2]}(0)=0$, with $p_1+p_2\neq 3$, and for $j=3,4$ $B_j(\lambda)y=y^{[p_j]}(a)+i\varepsilon_j\alpha\lambda y^{[q_j]}(a)=0$, with $p_j+q_j=3$ and $p_3\neq p_4$, and we investigate the asymptotics of the eigenvalues for $g=0$ and for arbitrary $g$. Note that, with the above considerations the operator $A(U)$ is self-adjoint if and only if $B_j(\lambda)y=y^{[p_j]}(a)+i\alpha\lambda y^{[q_j]}(a)=0$, while $q_j$ is odd and $B_j(\lambda)y=y^{[p_j]}(a)-i\alpha\lambda y^{[q_j]}(a)=0$ while $q_j$ is even for $j=3,4$. We classify the eigenvalue problems according the boundary conditions $B_1y=y^{[p_1]}(0)=0$ and $B_2y=y^{[p_2]}(0)=0$, with $p_1+p_2\neq 3$, thus we have four classes of eigenvalue problems, which are firstly the class where the boundary conditions at the endpoint 0 are $B_1y=y(0)=0$ and $B_2y=y''(0)=0$, the second class is the class where the boundary conditions at the endpoint 0 are $B_1y=y(0)=0$ and $B_2y=y'(0)$, followed by the class where the boundary conditions at the endpoint 0 are $B_1 y = y''(0)$ and $B_2 y = y^{[3]}(0)$ and finally the class where the boundary conditions at the endpoint 0 are $B_1 y = y'(0) = 0$ and $B_2 y = y^{[3]}(0) = 0$. In each of these classes we have the following four pairs of $\lambda$-dependent boundary conditions at the endpoint $a$, firstly we have the pair $y''(a) + i\alpha\lambda y'(a) = 0$ and $y^{[3]}(a) - i\alpha\lambda y(a) = 0$, secondly the pair $y''(a) + i\alpha\lambda y'(a) = 0$ and $y(a) + i\alpha\lambda y^{[3]}(a) = 0$, followed by the pair $y'(a) - i\alpha\lambda y''(a) = 0$ and $y(a) + i\alpha\lambda y^{[3]}(a) = 0$ and finally the pair $y'(a) - i\alpha\lambda y''(a) = 0$ and $y^{[3]}(a) - i\alpha\lambda y(a) = 0$. For the case $g = 0$, the boundary conditions $y^{[3]}(0) = 0$, $y(a) + i\alpha\lambda y^{[3]}(a) = 0$ and $y^{[3]}(a) - i\alpha\lambda y(a) = 0$ are respectively replaced by the boundary conditions $y^{(3)}(0) = 0$, $y(a) + i\alpha\lambda y^{(3)}(a) = 0$ and $y^{(3)}(a) - i\alpha\lambda y(a) = 0$. As the quadratic operator pencil (1.6) has self-adjoint operators as coefficients, then the eigenvalues of each of the eigenvalue problems (1.1)–(1.5), for which the operator $A(U)$ is self-adjoint, lie in the closed upper half-plane and on the imaginary axis, see [34]. We study during this research spectral properties of the eigenvalue problem (1.7)–(1.8). We prove that the operator pencil $L(\cdot, \alpha)$ is a Fredholm valued function with index 0 and the spectrum of $L(\cdot, \alpha)$ consists of discrete eigenvalues with finite multiplicities and all the eigenvalues of $L(\cdot, \alpha)$, $\alpha \geq 0$, lie in the closed upper half-plane and on the imaginary axis and are symmetric with respect to the imaginary axis, see Proposition 3.13. We investigate the semi-simplicity, the geometric and the algebraic multiplicities of the eigenvalues of the problem (1.7)–(1.8), see Lemma 3.14, Lemma 3.15, Proposition 3.16 and Lemma 3.17. We prove that all nonzero real eigenvalues of $L(\cdot, \alpha)$, $\alpha > 0$ and the pure imaginary eigenvalues with negative imaginary parts are semi-simple, see Lemma 3.14 and Lemma 3.15. We show that all real eigenvalues of $L(\cdot, \alpha)$ are independent of $\alpha$, see Lemma 3.14. To investigate the geometric and algebraic multiplicity of the eigenvalues of the problem (1.7)–(1.8), we use the characteristic determinant, eigenvectors and associated vectors corresponding to the respective eigenvalues. We justify the differentiability of the eigenvalues at $\alpha \geq 0$ and prove that $\Re\dot{\lambda}_k(\alpha) = 0$ and $\Im\dot{\lambda}_k(\alpha) \geq 0$ for $\lambda_k(\alpha) = i\tau$, where $\tau \in \mathbb{R} \setminus \{0\}$, see Lemma 3.17. We derive formulas for the asymptotics of the eigenvalues for $g = 0$, see Chapter 4. We use these formulas to develop the corresponding formulas for general $g$, see Chapter 5. We represent, in each case, the characteristic matrix for $g = 0$ by $(B_i y_j)_{i,j=1}^4$, where the $B_i y = 0$, $i = 1, 2, 3, 4$ are the boundary conditions defined in (1.8), and $y_j$, $j = 1, 2, 3, 4$, the fundamental system such that $y_j^{(m)} = \delta_{j,m+j}$, for $m = 0, 1, 2, 3$ for the case $g = 0$ and $y_j^{[m]} = \delta_{j,m+j}$, for $m = 0, 1, 2, 3$

for general $g$ and $\delta$ the Kronecker delta. We derive in each case, for $g = 0$, the characteristic equation from the above representation of the characteristic matrix. We carefully define the characteristic equation $\phi(\mu) = a_0(\alpha)\phi_0(\mu) + a_1(\alpha)\phi_1(\mu)$, where $\mu^2 = \lambda$, $a_0(\alpha)$ and $a_1(\alpha)$ depend or not on $\alpha$, $\phi_0(\mu)$ is the unperturbed part of $\phi(\mu)$, while $\phi_1(\mu)$ is its the perturbed part. We count all the zeros of $\phi_0(\mu)$ with their proper multiplicities. We give the asymptotics of the eigenvalues, with exact indexing for the zeros of the unperturbed function. We use Rouché's theorem to derive and count the asymptotics of the zeros of $\phi$. We prove that each of the sixteen eigenvalue problems is Birkhoff regular for $\alpha > 0$, see Proposition 5.31. We derive a fundamental system at order $o(\mu^{-4})$ for the differential equation $y^{(4)} - (gy')' = \lambda^2 y$, however we only use the resulting fundamental system at order $o(\mu^{-2})$ to give the asymptotics of the eigenvalues. We denote respectively by $D$ and $D_0$ the characteristic function, of each of the sixteen eigenvalue problems (1.7)–(1.8), for general $g$ and $g = 0$. Since each of the sixteen eigenvalue problems described above are Birkhoff regular, then $g$ influences only lower order terms in $D$, consequently higher order terms do not vanish from $D$, it follows that away from small rectangles around the zeros of $D_0$ and on the boundaries of large squares $S_k = \pm k\frac{\pi}{a} \pm ik\frac{\pi}{a}$, $k \in \mathbb{N}$, $|D(\mu) - D_0(\mu)| < |D_0(\mu)|$ if $|\mu|$ is sufficiently large. Therefore applying Rouché's theorem to the boundaries of small rectangles around the zeros of $D_0$ and to large squares $S_k$, it results in the eigenvalues of each of the eigenvalue problems (1.7)–(1.8) for general $g$ having the same asymptotics as the eigenvalues for $g = 0$. Because of the symmetry with respect to the imaginary axis of each of the eigenvalue problems, we investigate the eigenvalue asymptotics along the positive real axis only, where the first four terms of the eigenvalue asymptotics are derived. We use the computer algebra Maple for the computations involving the asymptotics of the eigenvalues for $g = 0$, while we use Sage for the calculations of the third and fourth terms of the eigenvalue asymptotics. A summary of the research conducted on these self-adjoint fourth-order eigenvalue problems with the above enumerated independent boundary conditions at the endpoint 0 and the following $\lambda$-dependent boundary conditions at the endpoint $a$: $y''(a) + i\alpha\lambda y'(a) = 0$ and $y^{[3]}(a) - i\alpha\lambda y(a) = 0$ can be found in [40].

The second part of our study consists of a sixth-order eigenvalue problem defined by the differential equation

$$-y^{(6)} - (gy'')'' = \lambda^2 y, \tag{1.13}$$

and the boundary conditions

$$y(\lambda, 0) = 0, \tag{1.14}$$

$$y'(\lambda, 0) = 0, \tag{1.15}$$

$$y''(\lambda, 0) = 0, \tag{1.16}$$

$$y(\lambda, a) = 0, \tag{1.17}$$

$$y''(\lambda, a) = 0, \tag{1.18}$$

$$y^{(4)}(\lambda, a) - i\alpha\lambda y'(\lambda, a) = 0, \tag{1.19}$$

where $g \in C^2[0, a]$, $a > 0$ and $\alpha > 0$. We associate a quadratic operator pencil

$$L(\lambda, \alpha) = \lambda^2 M - i\alpha\lambda K - A \tag{1.20}$$

in the space $L_2(0, a) \oplus \mathbb{C}$ with this problem, where $K$ and $M$ are bounded self-adjoint operators with domains $\mathscr{D}(K) = \mathscr{D}(M) = L_2(0, a) \oplus \mathbb{C}$, and given by

$$K = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \text{ and } M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.21}$$

The operator $A$ acting in $L_2(0, a) \oplus \mathbb{C}$ with domain

$$\mathscr{D}(A) = \left\{ \widetilde{y} = \begin{pmatrix} y \\ -y'(a) \end{pmatrix} : y \in W_6^2(0, a), y(0) = y'(0) = y''(0) = y(a) = y''(a) = 0 \right\}, \tag{1.22}$$

$$\text{is given by} \quad A\widetilde{y} = \begin{pmatrix} -y^{(6)} - (gy'')'' \\ y^{(4)}(a) \end{pmatrix} \text{ for } \widetilde{y} \in \mathscr{D}(A). \tag{1.23}$$

We prove in Section 6.2 that the operator $A$ is self-adjoint, thus the operator pencil $L(\cdot, \alpha)$ defined in (1.20) has self-adjoint operators as coefficients. Again, we use the approach of [34] to investigate the spectral asymptotics of the sixth-order eigenvalue problem (1.13)–(1.19). We show that the operator pencil $L(\cdot, \alpha)$ is a Fredholm valued function with index 0 and its spectrum consists of discrete eigenvalues of finite multiplicities and all the eigenvalues of $L(\cdot, \alpha)$, $\alpha \geq 0$, lie in the closed upper half-plane and on the imaginary axis and are symmetric with respect to the imaginary axis, see Proposition 6.23. We prove, respectively in Lemma 6.24 and in Lemma 6.25, that all nonzero real eigenvalues of $L(\cdot, \alpha)$, $\alpha > 0$ and the pure imaginary eigenvalues with negative imaginary parts are semi-simple. We show, in Lemma 6.24, that all real eigenvalues of $L(\cdot, \alpha)$, where $\alpha > 0$, are independent of $\alpha$. We prove in Proposition 6.26 that the geometric multiplicity of all the eigenvalues of $L(\cdot, \alpha)$ is at most 3.

To derive the asymptotics of the eigenvalues for $g = 0$, we represent the characteristic matrix by $(B_i y_j)_{i,j=1}^{6}$, where $B_i y = 0$, $i = 1, \ldots, 6$ are the boundary conditions (1.14)–(1.19), $y_j$, $j = 1, \ldots, 6$, the fundamental system such that $y_j^{(m)} = \delta_{j,m+j}$, for $m = 0, \ldots, 5$. We use the reduced characteristic matrix to derive the characteristic function of this eigenvalue problem. We denote by $D$ the characteristic function of the eigenvalue problem (1.13)–(1.19) and by $D_0$ the corresponding $D$ for $g = 0$. Applying Rouché's theorem to the boundaries of the small squares around the zeros of $D_0$, we derive the eigenvalues of the problem (1.13)–(1.19) for $g = 0$. The Birkhoff regularity of the problem makes that $g$ influences only lower order terms in $D$, it follows that the higher order terms do not vanish from $D$. Therefore away from small squares around the zeros of $D_0$, the characteristic functions $D_0$ and $D$ are analytic functions and higher order terms of the eigenvalue asymptotics for $g = 0$ and general $g$ are the same. We compute the first four terms of the eigenvalue asymptotics for general $g$. We use the computer algebra Maple to compute the characteristic function of the eigenvalue problem for $g = 0$, while we use Sage for the computations related to the eigenvalue problem for general $g$. Because of the complexity of the sixth-order problem, the outputs of the third and fourth terms of the eigenvalue asymptotics are lengthy, hence they cannot be decrypted, so we compute manually these numbers. Using the values of the first four terms of the eigenvalue asymptotics, we give more precise asymptotics of the eigenvalues $\hat{\mu}_{k,n}$ in the sectors $[-\frac{\pi}{4} + \frac{j\pi}{3}, \frac{\pi}{12} + \frac{j\pi}{3}]$, where $j = 0, 1, 2, 3, 4, 5$ and therefore more precise asymptotics of the eigenvalues $\hat{\lambda}$ of the problem (6.1)–(6.7) in Theorem 6.34.

It would be interesting to study the symmetry of the self-adjoint fourth-order problems investigated during this research with respect to the endpoints 0 and $a$. In other words replace the independent boundary conditions $B_j y = 0$, $j = 1, 2$ by $\lambda$-dependent boundary conditions at the endpoint 0 and the $\lambda$-dependent boundary conditions $B_j y = 0$, $j = 3, 4$ by independent boundary conditions at the endpoint a and investigate the asymptotics of the eigenvalues. On the other hand, it would also be interesting to study the self-adjoint fourth-order problems (1.7)–(1.8), firstly when only one of the boundary conditions $B_jy = 0$, $j = 3, 4$ is $\lambda$-dependent, secondly the boundary conditions $B_jy = 0$, $j = 3, 4$ are $\lambda$-dependent and one the boundary conditions $B_jy = 0$, $j = 1, 2$ is $\lambda$-dependent, thirdly all the boundary conditions $B_jy = 0$, $j = 1, 2, 3, 4$, are $\lambda$-dependent. It would be exciting to investigate the more general case where the six two-point boundary conditions of the sixth-order boundary problem defined in (1.13)–(1.19) may depend or not on $\lambda$ and give necessary and sufficient conditions for which the corresponding operator $A(U)$ is self-adjoint, where $U$ denotes the collection of the corresponding six boundary conditions. And for each of the cases where the corresponding operator $A(U)$ is self-adjoint, investigate the asymptotics of the eigenvalues. We plan to conduct gradually each of the above mentioned studies in a future time.

The structure of this document is the following, after the introductory chapter in which we give a summary of the work done during this research, we give in Chapter 2 some of the concepts and properties necessary for the comprehension of this document and to conduct the research. We present in Chapter 3 the spectral properties of the self-adjoint fourth-order boundary value problems that we have investigated during this study. We give in Chapter 4 notions and properties on Rouché's theorem and asymptotic fundamental matrices followed by the asymptotics of the eigenvalues for $g = 0$ of the each of the sixteen self-adjoint fourth-order eigenvalue problems investigated. We recall that Rouché's theorem and asymptotic fundamental matrices have been intensively used in Chapter 4. We present in Chapter 5 notions and properties of Birkhoff regular eigenvalue problems and an asymptotic fundamental system for the differential equation

$$K\eta = \eta^{(n)} + \sum_{i=0}^{n-1} k_i \eta^{(i)}, \tag{1.24}$$

$$H\eta = \sum_{i=0}^{n_0} h_i \eta^{(i)}, \tag{1.25}$$

with $0 \le n_0 \le n-1$ and $k_i, h_i \in W_i^2(a, b)$ with the assumptions $h_{n_0} > 0$, $h_{n_0}^{-1} \in L_\infty(a, b)$ and $h_{n_0}^{-1} \in W_1^2(a, b)$ if $n_0 > 0$ and for $n_0 = 0$, $h_{n_0}^{-1} \in W_1^2(a, b)$. We use the concepts and properties on Birkhoff regularity to show that each of the self-adjoint fourth-order boundary value problems is Birkhoff regular, while the concepts and properties on asymptotic fundamental system are used to derive an asymptotic fundamental system of the differential equation $y^{(4)} - (gy')' = \lambda^2 y$. Finally we give in Chapter 5 the asymptotics of the eigenvalues for general $g$ of each of the self-adjoint fourth-order boundary value problems investigated. All the results on the self-adjoint sixth-order eigenvalue problem investigated during this research are presented in Chapter 6. Firstly we use the concepts and properties of distributions and Sobolev spaces presented in Chapter 2 to show that the operator $A$ associated to the sixth-order problem is self-adjoint, secondly we study the spectral properties of this self-adjoint sixth-order problem, thirdly we investigate the asymptotics of the eigenvalues of the problem for $g = 0$ and finally we use the asymptotics of the eigenvalues for $g = 0$ to derive the asymptotics of the eigenvalues of the problem for general $g$. We give in the last chapter for the readers who are interested concepts and properties on exponential sums and the codes that we have developed during this research to compute the asymptotics of the eigenvalues for general $g$.

# Chapter 2

# Preliminaries

## 2.1 Introduction

We present in this chapter basic definitions and properties necessary to conduct and understand the work presented in this document. We give in Section 2.2 definitions and properties of linear operators. We provide in Section 2.3 definitions and properties of test functions in Subsection 2.3.1 followed by definitions and properties of distributions and Sobolev spaces in Subsection 2.3.2. We present in Section 2.4 definitions and properties of holomorphic and meromorphic vector valued functions, while we give in Section 2.5 notions and properties of abstract boundary eigenvalue operator functions followed by notions and properties of boundary eigenvalue problem in Section 2.6. These definitions and properties are extensively used to conduct the work that we present in this document. Other definitions and properties are presented in subsequent chapters where they are more relevant.

## 2.2 Linear differential operators

**Definition 2.1.** [41, page 3]. A linear differential expression is an expression of the form:

$$l(y) := p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y \tag{2.1}$$

where $\frac{1}{p_0(x)},\ p_1(x),\ p_2(x),\ \ldots\ ,p_n(x)$ are continuous functions on a fixed, finite, interval $[a,b]$ and $y \in C^n[a,b]$.

**Definition 2.2.** [41, page 3]. If linear combinations

$$\begin{aligned} B_j(y) &= \alpha_0^j y(a) + \alpha_1^j y'(a) + \cdots + \alpha_{n-1}^j y^{(n-1)}(a) \\ &\quad + \beta_0^j y(b) + \beta_1^j y'(b) + \cdots + \beta_{n-1}^j y^{(n-1)}(b), \qquad j = 1, \ldots, m \end{aligned} \tag{2.2}$$

of the values of the function $y$ and its first $n-1$ successive derivatives at the boundary points $a$ and $b$ of the interval $[a,b]$ have been specified and the conditions $B_j(y) = 0,\ j = 1, \ldots, m$, are imposed on the functions $y \in C^n[a,b]$, these conditions which the functions $y$ must satisfy are called boundary conditions.

Let $\tau$ be the formal differential expression of the form

$$\tau y(x) = r(x)^{-1}\left\{\sum_{j=0}^{[\frac{n}{2}]}(-1)^j (p_j(x)y^{(j)}(x))^{(j)} + \sum_{j=0}^{[\frac{n-1}{2}]}(-1)^j[(q_j(x)y^{(j)}(x))^{(j+1)} - (q_j^*(x)y^{(j+1)}(x))^{(j)}]\right\}, \tag{2.3}$$

where

- $y$ are $C^n$-valued functions defined on $(a,b)$, $-\infty \le a < b \le \infty$,
- the symbol $[\alpha]$ stands for the largest integer less than or equal to $\alpha$,
- the coefficients $r,\ p_j$ and $q_j$ are $m \times m$ matrix valued functions on $(a,b)$, $r(x)$ is positive definite, the $p_j(x)$ are Hermitian and $q_j^*$ is the adjoint of $q_j$.

**Definition 2.3.** [51, pages 26-27]. Let the above hypotheses hold. We define the quasi-derivatives for $k = 1$ $(n = 2)$:

$$\begin{aligned} y^{[0]} &= y, \\ y^{[1]} &= p_1 \frac{d}{dx} y^{[0]} - q_0 y^{[0]}, \\ y^{[2]} &= -\frac{d}{dx} y^{[1]} - q_0^* p_1^{-1}(y^{[1]} + q_0 y^{[0]}) + p_0 y^{[0]} \\ &= -\frac{d}{dx} y^{[1]} - q_0^* p_1^{-1} y^{[1]} + (p_0 - q_0^* p_1^{-1} q_0) y^{[0]} = rly, \end{aligned}$$

and for $n = 2k$ and $k \geq 2$:

$$\begin{aligned}
y^{[j]} &= y^{(j)} = \frac{d^j}{dx^j}y \quad \text{for} \quad j = 0, \dots, k-1, \\
y^{[k]} &= p_k \frac{d}{dx} y^{[k-1]} - q_{k-1} y^{[k-1]}, \\
y^{[k+1]} &= -\frac{d}{dx} y^{[k]} + p_{k-1} y^{[k-1]} - q^*_{k-1} \frac{d}{dx} y^{[k-1]} - q_{k-2} y^{[k-2]} \\
&= -\frac{d}{dx} y^{[k]} - q^*_{k-1} p_k^{-1} y^{[k]} + [p_{k-1} - q^*_{k-1} p_k^{-1} q_{k-1}] y^{[k-1]} - q_{k-2} y^{[k-2]}, \\
y^{[k+j]} &= -\frac{d}{dx} y^{[k+j-1]} + p_{k-j} y^{[k-j]} - q^*_{k-j} y^{[k-j+1]} - q_{k-j-1} y^{[k-j-1]} \quad \text{for} \quad j = 2, \dots, k-1, \\
y^{[n]} &= y^{[2k]} = -\frac{d}{dx} y^{[n-1]} + p_0 y - q^*_0 y^{[1]} = rly.
\end{aligned}$$

**Proposition 2.4.** [42, page 18]. *If the coefficients $p_k(x)$, $k = 0, \dots, n$ of the differential expression $l(y)$, see* (2.1)*, have continuous derivatives up to the order $(n-k)$ inclusive on the interval $[a, b]$, then there exists a differential expression $l^*(z)$, where $z \in C^{(n)}[a, b]$ such that*

$$\int_a^b l(y)\bar{z}dx = \int_a^b y\overline{l^*(z)}dx + [y, z]_a^b, \tag{2.4}$$

*where*

1. $$[y, z] = \sum_{k=1}^{n} (y^{[k-1]} \bar{z}^{[2n-k]} - y^{[2n-k]} \bar{z}^{[k-1]}) \tag{2.5}$$

   *is Lagrange's form,*

2. $$[y, z]_a^b \tag{2.6}$$

   *is the difference in the values for the function $[y, z]$, defined in* (2.5)*, for $x = b$ and $x = a$,*

3. (2.4) *is said to be Lagrange's identity in integral form.*

**Definition 2.5.** [41, page 7]. The differential expression $l^*(z)$, defined in (2.4) is called the adjoint of the differential expression $l(y)$.

**Definition 2.6.** If $l = l^*$, then $l(y)$ is said to be formally self-adjoint. See [42, page 50].

**Proposition 2.7.** [41, page 8]. *Any formally self-adjoint differential expression with real coefficients is necessarily of even order and has the form*

$$l(y) = (p_0 y^{(\mu)})^{(\mu)} + (p_1 y^{(\mu-1)})^{(\mu-1)} + \cdots + (p_{\mu-1} y')' + p_\mu y.$$

**Definition 2.8.** [21, page 164]. Let $\mathscr{H}$, $\mathscr{H}_1$ be two Hilbert spaces, and $T$ be an operator from $\mathscr{H}$ to $\mathscr{H}_1$ .

A sequence $u_n \in \mathscr{D}(T)$ will be said to be $T$-convergent (to $u \in \mathscr{H}$) if both $\{u_n\}$ and $\{Tu_n\}$ are Cauchy sequences and $u_n \to u$. We shall write $u_n \underset{T}{\to} u$ to denote that $\{u_n\}$ is $T$-convergent to $u$.

$T$ is said to be closed if $u_n \underset{T}{\to} u$ implies $u \in \mathscr{D}(T)$ and $Tu = \lim Tu_n$; in other words if, for any sequence $u_n \in \mathscr{D}(T)$ such that $u_n \to u$ and $Tu_n \to v$, $u$ belongs to $\mathscr{D}(T)$ and $Tu = v$. The set of all closed operators from $\mathscr{H}$ to $\mathscr{H}_1$ is denoted by $\mathscr{C}(\mathscr{H}, \mathscr{H}_1)$.

**Definition 2.9.** Let $T$ be a linear operator from a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{H}_1$. We can define a special inner product and norm on $\mathscr{H}$ by

$$(u, v)_T = (u, v) + (Tu, Tv),$$
$$||u||_T = (u, u)_T^{1/2} = [||u||^2 + ||Tu||^2]^{1/2}.$$

We refer to this inner product structure as the graph norm structure on $\mathscr{D}(T)$. See Remark 2.3 of [29, page 6].

**Definition 2.10.** [29, page 6]. Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, and $T$ be a linear operator in $\mathscr{H}$ to $\mathscr{H}_1$. Then the graph of $T$ is the subset $\Gamma(T)$ of $\mathscr{H} \times \mathscr{H}_1$ defined by

$$\Gamma(T) = \{[u, Tu] \quad | \quad u \in \mathscr{D}(T)\}.$$

**Proposition 2.11.** [29, page 6]. *Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces and $T$ be a linear operator from $\mathscr{H}$ to $\mathscr{H}_1$. Then $\Gamma(T)$ is a subspace of $\mathscr{H} \times \mathscr{H}_1$. The inverse graph of $T$ is the subspace of $\mathscr{H}_1 \times \mathscr{H}$ defined by*

$$\Gamma'(T) = \{[Tu, u] \quad | \quad u \in \mathscr{D}(T)\}.$$

**Proposition 2.12.** *Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, and $T$ be a linear operator from $\mathscr{H}$ to $\mathscr{H}_1$.*

*$T$ is closed if and only if the graph $\Gamma'(T)$ is closed in $\mathscr{H}_1 \times \mathscr{H}$.* See Remark 2.2 of [29, page 6].

**Proposition 2.13.** [29, page 6]. *Let $T$ be a linear operator from a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{H}_1$.*

- *$T$ is continuous from the graph norm structure on $\mathscr{D}(T)$ into the standard structure on $\mathscr{H}_1$.*
- *$\mathscr{D}(T)$ is a Hilbert space under the graph norm structure if and only if $T$ is closed.*

**Remark 2.14.** [42, page 9] . Let $T$ be a linear operator in a Hilbert space $\mathscr{H}$. If $T$ is not closed, then by definition its graph $\Gamma(T)$ is not closed in $\mathscr{H} \times \mathscr{H}$.

**Remark 2.15.** [42, page 9]. Let $T$ be a linear operator from a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{H}_1$. If $T$ is not closed, then it may happen that the closure $\overline{\Gamma}_T$ of the graph $\Gamma(T)$ in $\mathscr{H} \times \mathscr{H}_1$ is the graph of certain operator. In such a case this operator is called the closure of the operator $T$. We then say that the operator $T$ admits a closure denoted by $\overline{T}$. Thus by definition $\Gamma_{\overline{T}} = \overline{\Gamma}_T$.

**Proposition 2.16.** [42, page 10]. *Let $\mathscr{H}$ be a Hilbert space. A subspace $S$ lies dense in $\mathscr{H}$ if and only if there is no nonzero vector in $\mathscr{H}$ which is orthogonal to $S$.*

**Definition 2.17.** If the domain $\mathscr{D}(L)$ of a linear operator $L$ is *dense* in a Hilbert space $\mathscr{H}$, then $L$ is said to be densely defined. If $L$ is densely defined and its range is contained in a Hilbert space $\mathscr{H}_1$, the mapping $L^*$, the adjoint of $L$ has as domain

$$D(L^*) = \{z \in \mathscr{H}_1 : \exists\ w \in \mathscr{H} \quad \langle y, w\rangle = \langle Ly, z\rangle \quad \forall y \in \mathscr{D}(L)\}.$$

See Definition 2.7.4 of [20, page 157].

**Definition 2.18.** [42, page 13]. Let $T : \mathscr{H} \to \mathscr{H}$ be a linear operator, where $\mathscr{H}$ is a Hilbert space. Then

- $T$ is said to be Hermitian if for all $y,\ z \in \mathscr{D}(T)$ $\langle Ty, z\rangle = \langle y, Tz\rangle$ holds.
- A Hermitian operator which is densely defined in a Hilbert space $\mathscr{H}$ is called symmetric operator.

- A symmetric operator $T$ in a Hilbert space $\mathscr{H}$ is said to be self-adjoint if $T = T^*$.

**Definition 2.19.** [26, pages 4-5]. Let $L(\mathscr{H}_1, \mathscr{H}_2)$ be the space of linear operator from $\mathscr{H}_1$ into $\mathscr{H}_2$, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces and $\mathscr{H}_1$ is dense in $\mathscr{H}_2$. An operator *pencil* is a polynomial $\mathcal{A}(\lambda) = \sum\limits_{q=0}^{l} A_{l-q}\lambda^q$ in $\lambda \in \mathbb{C}$ where $A_q \in L(\mathscr{H}_1, \mathscr{H}_2)$. If the equation $\mathcal{A}(\lambda_0)y = 0$ has nontrivial solutions, then $\lambda_0$ is called an *eigenvalue* of the operator pencil $\mathcal{A}$, and the corresponding nontrivial solutions are called *eigenvectors* related to $\lambda_0$.

**Definition 2.20.** [21, page 142]. Let $T : \mathscr{H} \to \mathscr{H}_1$ be a linear operator, where $\mathscr{H}$ and $\mathscr{H}_1$ are Hilbert spaces. Let $D(T)$ be the domain of definition of $T$.
The range $R(T)$ of $T$ is the set of all vectors of the form $Tu$ with $u \in D(T)$.
The null space $N(T)$ of $T$ is the set of all vectors $u \in D(T)$ such that $Tu = 0$.

**Definition 2.21.** [21, pages 229-230]. Let $T$ be a densely defined operator from the Hilbert space $\mathscr{H}$ to be Hilbert space $\mathscr{H}_1$. Then

$$\begin{aligned}
&\operatorname{nul}(T) = \dim N(T),\\
&\operatorname{def}(T) = \dim N(T^*) = \dim R(T)^{\perp},\\
&\operatorname{ind}(T) = \operatorname{nul}(T) - \operatorname{def}(T) \text{ if at least one of } \operatorname{nul}(T) \text{ and } \operatorname{def}(T) \text{ is finite.}
\end{aligned}$$

The integers $\operatorname{nul}(T)$, $\operatorname{def}(T)$, and $\operatorname{ind}(T)$ are called the nullity, deficiency, and index, respectively, of the operator $T$.

**Definition 2.22.** [21, page 230]. Let $\mathscr{H}$ and $\mathscr{H}_1$ be two Hilbert spaces. Let $T \in \mathscr{C}(\mathscr{H}, \mathscr{H}_1)$ and $T$ be densely defined. Then $T$ is said to be *semi-Fredholm* if $R(T)$ is closed and at least one of the $\operatorname{nul}(T)$ and $\operatorname{def}(T)$ is finite.

**Definition 2.23.** [21, page 230]. Let $T$ be a linear operator from a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{H}_1$. $T$ is a *Fredholm operator* if it satisfies the following conditions:

1. $T \in \mathscr{C}(\mathscr{H}, \mathscr{H}_1)$,
2. the range $R(T)$ is closed in $\mathscr{H}_1$,
3. the null spaces $N(T)$ and $N(T^*) = R(T)^{\perp}$ are both finite.

**Proposition 2.24.** *Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, $T \in \mathscr{C}(\mathscr{H}, \mathscr{H}_1)$ and $T$ be densely defined. $T$ is Fredholm (semi-Fredholm) if and only if $T^*$ is. In this case* $\operatorname{ind} T^* = -\operatorname{ind} T$. *If $T$ is a Fredholm operator from $\mathscr{H}$ to $\mathscr{H}_1$, then* $\operatorname{nul}(T) = \operatorname{def}(T^*)$ *and* $\operatorname{def}(T) = \operatorname{nul}(T^*)$.

See Corollary 5.14 [21, page 234].

**Remark 2.25.** [31, page 2]. Let $\mathscr{H}$ and $\mathscr{H}_1$ be two Hilbert spaces. We denote the space of Fredholm operators in $L(\mathscr{H}, \mathscr{H}_1)$ by $\Phi(\mathscr{H}, \mathscr{H}_1)$.

**Definition 2.26.** [21, page 231]. Let $T$ be defined as in Proposition 2.24. Let $\widetilde{\mathscr{H}}$ be the quotient space $\mathscr{H}/N(T)$. We can define an operator $\widetilde{T}$ from $\widetilde{\mathscr{H}}$ to $\mathscr{H}_1$ by

$$\widetilde{T}\widetilde{u} = Tu,$$

where the domain $\mathscr{D}(\widetilde{T})$ of $\widetilde{T}$ is the set of all $\widetilde{u} \in \widetilde{\mathscr{H}}$ such that every $u \in \widetilde{u}$ belongs to $\mathscr{D}(T)$. If $u \in \mathscr{D}(T)$, all $u' \in \widetilde{u}$ belong to $\mathscr{D}(T)$ by $u' - u \in N(T) \subset \mathscr{D}(T)$.
The minimum modulus of $T$ is the number $\gamma(T) = \frac{1}{||\widetilde{T}^{-1}||}$.

**Remark 2.27.** [21, page 231]. It is to be understood that that $\gamma(T) = 0$ if $\widetilde{T}^{-1}$ is unbounded and $\gamma(T) = \infty$ if $\widetilde{T}^{-1} = 0$.

**Definition 2.28.** [21, page 190]. Let $\mathscr{H}$ be a Hilbert space. Let $T$ and $A$ be operators with the same domain space $\mathscr{H}$ (but not necessarily with the same range space) such that $\mathscr{D}(T) \subset \mathscr{D}(A)$ and

$$||Au|| \leq a||u|| + b||Tu||, \quad u \in \mathscr{D}(T), \tag{2.7}$$

where $a$, $b$ are nonnegative constants. The operator $A$ is said to be relatively bounded with respect to $T$ or simply $T$-bounded.

**Theorem 2.29.** *Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces. Let $T \in \mathscr{C}(\mathscr{H}, \mathscr{H}_1)$ be semi-Fredholm (so that $\gamma = \gamma(T) > 0$). Let $A$ be a $T$-bounded operator from $\mathscr{H}$ to $\mathscr{H}_1$ so that we have the inequality* (2.7)*, where*

$$a < (1 - b)\gamma. \tag{2.8}$$

*Then $S = T + A$ belongs to $\mathscr{C}(\mathscr{H}, \mathscr{H}_1)$, $S$ is semi-Fredholm and*

$$\operatorname{nul} S \leq \operatorname{nul} T, \quad \operatorname{def} S \leq \operatorname{def} T, \quad \operatorname{ind} S = \operatorname{ind} T. \tag{2.9}$$

See Theorem 5.22. [21, page 236].

## 2.3 Sobolev Spaces on Intervals

We assume in this section that $a$ and $b$ are real numbers with $a < b$. Let $1 \leq p \leq \infty$.

### 2.3.1 Test Functions and Distributions

**Definition 2.30.** [31, pages 53, 54]. Let $I \subset \mathbb{R}$ be an interval.

- $C(I) = C^0(I)$ denotes the space of all continuous functions on $I$ to $\mathbb{C}$. For a positive integer $k$, $C^k(I)$ denotes the space of $k$-times continuously differentiable functions on $I$.
- For $f \in C(I)$ the set $\operatorname{supp} f := \overline{\{x \in I : f(x) \neq 0\}}$ is called the support of $f$, where the closure is taken with respect to $I$.
- Let $I$ be open and let $C^\infty(I) := \bigcap_{k=1}^{\infty} C^k(I)$. A function $f \in C^\infty(I)$ is called a test function if its support is a compact subset of $I$. The space of all test functions on an open interval $I$ is denoted by $C_0^\infty(I)$.

**Remark 2.31.** The space $C_0^\infty(I)$ can be identified with a subset of $C_0^\infty(\mathbb{R})$ by setting $f = 0$ outside of $I$ for each $f \in C_0^\infty(I)$. The space $C_0^\infty(I)$ can be written as $C_0^\infty(I) = \bigcup_{K \subset I,\ compact} C_0^\infty(K)$, where $C_0^\infty(K) := \{f \in C_0^\infty(\mathbb{R}) : \operatorname{supp} f \subset K\}$.

**Definition 2.32.** [31, page 54]. Let $I$ be an open interval. A linear functional $u$ on $C_0^\infty(I)$ is called a distribution on $I$ if for each compact set $K \subset I$ there are numbers $k \in \mathbb{N}$ and $C \geq 0$ such that

$$|\langle \phi, u \rangle| \leq C \sum_{j=0}^{k} \sup_{x \in K} |\phi^{(j)}(x)| \qquad (\phi \in C_0^\infty(K)),$$

where $\langle \phi, u \rangle := u(\phi)$. The space of distribution is denoted by $\mathscr{D}'(I)$.

**Definition 2.33.** [31, page 55]. For $u \in \mathscr{D}'(I)$, where $I$ is an open interval, the support of $u$, denoted $\operatorname{supp} u$, is the set of points $x \in I$ such that for each neighborhood $U \subset I$ of $x$ there exists $\phi \in C_0^\infty(U)$ such that $\langle \phi, u \rangle \neq 0$.

**Definition 2.34.** [31, page 55]. Let $I$ be an open interval and let $u \in \mathscr{D}'(I)$. Then

$$\langle \phi, u' \rangle = -\langle \phi', u \rangle \quad (\phi \in C_0^\infty(I))$$

defines a distribution $u'$ on $I$, called the derivative in the sense of distributions of $u$. Recursively for $k = 1, 2, \ldots$

$$u^{(k+1)} := u^{(k)\prime}.$$

**Theorem 2.35.** *If $u \in \mathscr{D}'(X)$ where $X$ is an open interval on $\mathbb{R}$ and if $u' = 0$, then $u$ is a constant.*

See Theorem 3.1.4. of [17, page 57].

### 2.3.2 Definitions and Properties of Sobolev spaces

**Definition 2.36.** Let $I \subset \mathbb{R}$ be an open interval, $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. The space

$$W_k^p(I) := \{f \in L_p(I) : \ \forall j \in \{1, \ldots, k\} \ f^{(j)} \in L_p(I)\}$$

is called a Sobolev space. Here the derivatives $f^{(j)}$ are the derivatives in sense of distributions. For $f \in W_k^p(I)$ we set

$$|f|_{p,k} := \sum_{j=0}^{k} |f^{(j)}|_p.$$

Note that $W_0^p(I) = L_p(I)$ and that $L_2(I)$ is a Hilbert space with respect to the inner product

$$(f, g) = \int_I f(x)\bar{g}(x)dx, \ \ f, g \in L_2(I).$$

See Definition 2.1.1. of [31, page 55].

**Remark 2.37.** Let $I$ be an open interval. Let $AC^{loc}(I)$ be the set of functions $f$ on $I$ such that $f|_K$ is absolutely continuous for each compact subinterval $K$ of $I$. Then for $k > 0$, $W_k^p(I) = \{f \in AC^{loc}(I) : \ \forall j \in \{1, \ldots, k-1\} \ f^{(j)} \in AC^{loc}(I) \cap L_p(I), \ f^{(k)} \in L_p(I)\}$.

See Remark 2.1.2. of [31, page 55].

**Proposition 2.38.** *Let $I \subset \mathbb{R}$ be an open interval, $\gamma \in \bar{I}$ and $g \in L_p(I)$.Set*

$$G(x) := \int_\gamma^x g(t)dt \quad (x \in \bar{I}).$$

*Then $G$ is continuous on $\bar{I}$ and $G' = g$ in $\mathscr{D}'(I)$.*

See Proposition 2.1.3. of [31, page 56].

**Corollary 2.39.** *Let $k \in \mathbb{N}$ and $u \in \mathscr{D}'(a,b)$ such that $u' \in W_k^p(a,b)$. Then $u \in W_{k+1}^p(a,b)$.*

See Corollary 2.1.4. of [31, page 56].

**Proposition 2.40.** *Let $I \subset \mathbb{R}$ be an open interval and $k \in \mathbb{N}\backslash\{0\}$.*

1. *Let $f \in L_2(I)$ and $\gamma \in \bar{I}$. Then $f \in W_k^2(I)$ if and only if there are $g \in W_{k-1}^2(I)$ and $c \in \mathbb{C}$ such that*
$$f(x) = c + \int_\gamma^x g(t)dt \quad (x \in I).$$
*In this case, $g = f'$, $f$ has continuous extension to $\bar{I}$, which we also denote by $f$, and $c = f(\gamma)$.*

2. *$W_k^2(I) \subset C^{k-1}(\bar{I})$.*

See Proposition 2.1.5. of [31, page 56].

**Proposition 2.41.** *Let $I \subset \mathbb{R}$ be an open interval and $k \in \mathbb{N}$. Then $W_k^2(I)$ is a Hilbert space with respect to the norm $|\ \ |_{2,k}$.*

See Proposition 2.1.6. of [31, page 57].

**Proposition 2.42.** *For each $k \in \mathbb{N}$ we have*

1. *$W_{k+1}^2(a,b) \subset C^k[a,b]$,*

2. *$C^k[a,b] \subset W_k^2(a,b)$,*

*where the inclusions holds topologically, i.e., the corresponding inclusion maps are continuous.*

See Proposition 2.1.7. of [31, page 57].

**Proposition 2.43.** *Let $k, n \in \mathbb{N}$, $p_i \in W_{k+i}^2(a,b)$ for $i = 0, \dots, n$, $p_n^{-1} \in L_2(a,b)$, $\xi \in L_2(a,b)$. If $k = 0$, we additionally require that $p_0\xi \in L_1(a,b)$. Assume that*
$$\sum_{i=0}^n (p_i\xi)^{(i)} \in W_k^2(a,b). \tag{2.10}$$
*Then $\xi \in W_{k+n}^2(a,b)$.*

See Proposition 2.6.1. [31, page 74].

## 2.4 Holomorphic and meromorphic vector valued functions

We assume in this section that $\Omega$ is an open nonempty subset of $\mathbb{C}$, $a$ and $b$ are real numbers with $a < b$.

**Definition 2.44.** Let $\mathscr{H}$ be a Hilbert space, $y : \Omega \to \mathscr{H}$, $\lambda_0 \in \Omega$. The vector function $y$ is called holomorphic at $\lambda_0$ if there are a number $r > 0$ and $y_n \in \mathscr{H} \quad (n \in \mathbb{N})$ such that $K_r(\lambda_0) := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < r\} \subset \Omega$,

$$\sum_{n=0}^{\infty} r^n |y_n| < \infty, \tag{2.11}$$

and

$$y(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n y_n \tag{2.12}$$

for all $\lambda \in K_r(\lambda_0)$. The vector $y$ is called holomorphic in $\Omega$ if it is holomorphic at each $\lambda \in \Omega$. $H(\Omega, \mathscr{H})$ denotes the space of all holomorphic functions from $\Omega$ to $\mathscr{H}$.

See Definition 1.2.1 of [31, page 6].

**Proposition 2.45.** *A function $f : \Omega \to \mathscr{H}$ is holomorphic if and only if it is continuously differentiable.*

See Remark 1.2.2 of [31, page 6].

**Definition 2.46.** [38, page 336]. Let $\Omega$ be an open nonempty subset of $\mathbb{C}$, $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces. Let $\mathscr{L}(\mathscr{H}, \mathscr{H}_1)$ be the space of bounded linear operators from $\mathscr{H}$ to $\mathscr{H}_1$. An operator function $S : \Omega \to \mathscr{L}(\mathscr{H}, \mathscr{H}_1)$ is called holomorphic in $\Omega$ if for each $\lambda_0 \in \Omega$ it has a representation

$$S(\lambda) = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j S_j \tag{2.13}$$

which converges in the norm of $\mathscr{L}(\mathscr{H}, \mathscr{H}_1)$ for $\lambda$ in a neighborhood of $\lambda_0$.

**Proposition 2.47.** [31, page 3]. *Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces and $T$ a linear operator from $\mathscr{H}$ to $\mathscr{H}_1$. Then $T$ is invertible if and only if $T^*$ is invertible, and $(T^{-1})^* = (T^*)^{-1}$.*

**Definition 2.48.** [31, page 7]. Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces. For $T \in H(\Omega, \mathscr{L}(\mathscr{H}, \mathscr{H}_1))$,

$$\rho(T) := \{\lambda \in \Omega : T(\lambda) \text{ invertible}\}$$

is called the resolvent set of $T$ and $\sigma(T) := \Omega \setminus \rho(T)$ the spectrum of $T$. Let $T^{-1}(\lambda)$ be defined by $T^{-1}(\lambda) := T(\lambda)^{-1}$ for $\lambda \in \rho(T)$. The operator function $T^{-1}$ is called the resolvent of $T$.

**Proposition 2.49.** *Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces and $T \in H(\Omega, \mathscr{L}(\mathscr{H}, \mathscr{H}_1))$. Then $\rho(T)$ is open and $T^{-1} \in H(\rho(T), \mathscr{L}(\mathscr{H}_1, \mathscr{H}))$.*

See Proposition 1.2.5 of [31, page 7].

**Proposition 2.50.** *Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces and $T \in H(\Omega, \mathscr{L}(\mathscr{H}, \mathscr{H}_1))$. Set $T^*(\overline{\lambda}) := (T(\lambda))^*$ for $\lambda \in \Omega$. Then $T^* \in H(\Omega, \mathscr{L}(\mathscr{H}_1, \mathscr{H}))$.*

See Proposition 1.2.6 of [31, page 7].

**Definition 2.51.** [31, page 8]. Let $\mathscr{H}$ be a Hilbert space, $\mu \in \mathbb{C}$ and $y \in H(U' \setminus \{\mu\}, \mathscr{H})$ for some open neighborhood $U'$ of $\mu$ . We say that $y$ is meromorphic at $\mu$ if there is a nonnegative integer $s$ such that $(\cdot - \mu)^s y$ has a holomorphic continuation to all of $U'$. The smallest number $s$ is called the pole order of $y$ at $\mu$, and $\mu$ is called a pole of $y$ if this number is positive.

**Proposition 2.52.** [31, page 8]. *Let $\mathscr{H}$ be a Hilbert space, $\mu \in \mathbb{C}$ and $y \in H(U' \setminus \{\mu\}, \mathscr{H})$ for some open neighborhood $U'$ of $\mu$ . Then $y$ has a holomorphic extension to $\mu$ if and only if the pole order is zero.*

**Remark 2.53.** [31, page 8]. Let $\mathscr{H}$ be a Hilbert space, $\mu \in \mathbb{C}$ and $y \in H(U' \setminus \{\mu\}, \mathscr{H})$ for some open neighborhood $U'$ of $\mu$ . If $(\cdot - \mu)^s y$ is holomorphic at $\mu$, then the Laurent series expansion

$$y = \sum_{j=-s}^{\infty} (\cdot - \mu)^j y_j$$

holds in some punctured neighbourhood of $\mu$. We call

$$\sum_{j=-s}^{-1} (\cdot - \mu)^j y_j$$

the principal part of $y$ at $\mu$.

**Definition 2.54.** [31, page 9]. Let $U$ be an open subset of $\Omega$ and $y \in H(U, \mathscr{H})$. We say that $y$ is meromorphic in $\Omega$ if $\Omega \setminus U$ is a discrete subset of $\Omega$ and $y$ is meromorphic at each $\mu$ in $\Omega \setminus U$.

**Theorem 2.55.** *Let $\Omega$ be a domain in $\mathbb{C}$. Let $\mathscr{H}$ and $\mathscr{H}_1$ be two Hilbert spaces. Let $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$ and assume that $\rho(T) \neq \emptyset$. Then $\sigma(T)$ is a discrete subset of $\Omega$ and $T^{-1}$ is a meromorphic operator function in $\Omega$.*

See Theorem 1.3.1. [31, page 9].

**Definition 2.56.** Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$ and $\mu \in \Omega$. The vector function $y$ in $H(\Omega, \mathscr{H})$ is called root function of $T$ at $\mu$ if $y(\mu) \neq 0$ and $T(\mu)y = 0$. The number $\nu(y)$ denotes the order of the zero of $Ty$ at $\mu$ and is called the multiplicity of $y$ (with respect to $T$ at $\mu$).

See Definition 1.4.1 of [31, page 13].

**Definition 2.57.** [31, page 14]. Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$ such that $\rho(T) \neq \emptyset$. Let $\mu \in \sigma(T)$ and $n \in \mathbb{N} \setminus \{0\}$. Then $\widetilde{L}$ denotes the set of all $y_0 \in N(T(\mu))$ such that there is a root function $y$ with $y(\mu) = y_0$ and $\nu(y) \geq n$.

$$L_n := \widetilde{L}_n \cup \{0\}$$

is a subspace of $N(T(\mu))$.

**Definition 2.58.** [31, pages 14-15]. Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$ such that $\rho(T) \neq \emptyset$. Let $\mu \in \sigma(T)$ and $n \in \mathbb{N} \setminus \{0\}$. For $j \in \mathbb{N}$ with $0 < j \leq \operatorname{nul} T(\mu)$ we define

$$m_j := \max\{n \in \mathbb{N} \setminus \{0\} : \dim\ L_n \geq j\}. \tag{2.14}$$

The numbers $m_j$ are called the partial multiplicities of $T$ at $\mu$. The number $r = \dim\ N(T(\mu))$ is called the geometric multiplicity of $T$ at $\mu$, and the number

$$m = \sum_{j=1}^{r} m_j$$

is called the algebraic multiplicity of $T$ at $\mu$.

**Proposition 2.59.** *Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$, assume that $\rho(T) \neq \emptyset$ and let $\mu \in \sigma(T)$. Let $0 < r \leq \operatorname{nul} T(\mu)$ and let $y_1, \ldots, y_r$ be the root functions of $T$ at $\mu$ such that $y_1(\mu), \ldots, y_r(\mu)$ are linearly independent. The following conditions are equivalent:*

i) $\nu(y_j) = \max\{\nu(y) : y \textit{ is root function of } T \textit{ at } \mu \textit{ and } y(\mu) \notin \operatorname{span}\{y_1(\mu), \ldots, y_r(\mu)\}\}$ $(j = 1, \ldots, r)$,

ii) $\nu(y_j) = m_j \quad (j = 1, \ldots, r)$,

iii) $\nu(y_j) \geq m_j \quad (j = 1, \ldots, r)$.

See Proposition 1.4.4. [31, page 15].

**Definition 2.60.** Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$, $\rho(T) \neq \emptyset$ and $\mu \in \sigma(T)$. A system $\{y_1, \ldots, y_r\}$ of root functions of $T$ at $\mu$ is called a canonical system of root functions (CSRF) if $\{y_1(\mu), \ldots, y_r(\mu)\}$ is a basis of $N(T(\mu))$ and one of the equivalent conditions i), ii) or iii) in Proposition 2.59 is fulfilled.

See Definition 1.4.5. [31, page 15].

**Proposition 2.61.** *Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$ such that $\rho(T) \neq \emptyset$ and $\mu \in \sigma(T)$. We set $r := \operatorname{nul} T(\mu)$ and let $k \in \{0, \ldots, r-1\}$. Let $y_1(\mu), \ldots, y_k(\mu)$ be root functions of $T$ at $\mu$ with $\nu(y_j) \geq m_j$ for $j = 1, \ldots, k$ such that $y_1(\mu), \ldots, y_k(\mu)$ are linearly independent, where the numbers $m_j$ are partial multiplicities defined in* (2.14)*. Then there are root functions $y_{k+1}, \ldots . y_r$ of $T$ at $\mu$ such that $y_1, \ldots, y_r$ is a canonical system of root functions of $T$ at $\mu$.*

See Proposition 1.4.6. [31, page 15].

**Definition 2.62.** Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$ such that $\rho(T) \neq \emptyset$. Let $\mu \in \sigma(T)$.

i) An ordered set $\{y_0, y_1, \ldots, y_h\}$ in $\mathscr{H}$ is called chain of an eigenvector and associated vectors (CEAV) of $T$ at $\mu$ if

$$y := \sum_{l=0}^{h} (\cdot - \mu)^l y_l$$

is a root function of $T$ at $\mu$ with $\nu(y) \geq h+1$.

ii) Let $y_0 \in N(T(\mu)) \setminus \{0\}$. Then $\overline{\nu}(y_0)$ denotes the maximum of all multiplicities $\nu(y)$, where $y$ is a root function of $T$ at $\mu$ with $y(\mu) = y_0$. The number $\overline{\nu}(y_0)$ is called the rank of the eigenvector $y_0$.

iii) A system $\{y_l^{(j)} : 1 \leq j \leq r, 0 \leq l \leq \overline{m}_j - 1\}$ is called a canonical system of eigenvectors and associated vectors (CSEAV) of $T$ at $\mu$ if

$$\begin{aligned}
&\{y_0^{(1)}, \dots, y_0^{(r)}\} \quad \text{is a basis of} \quad N(T(\mu)),\\
&\{y_0^{(j)}, y_1^{(j)}, \dots, y_{\overline{m}_{j-1}}^{(j)}\} \quad \text{is a CEAV of T at } \mu \quad (\text{j} = 1, \dots, \text{r}),\\
&\overline{m}_j = \max\{\overline{\nu}(y) : y \in N(T(\mu)) \setminus \operatorname{span}\{y_0^{(k)} : k < j\}\} \quad (j = 1, \dots, r).
\end{aligned}$$

See Definition 1.6.1 [31, page 27].

**Proposition 2.63.** Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$ such that $\rho(T) \neq \emptyset$ and $\mu \in \sigma(T)$. Let $\{y_1, \dots, y_r\}$ be a CSRF of $T$ at $\mu$. Set

$$y_l^{(j)} := \frac{1}{l!} \frac{d^l y_j}{d\lambda^l}(\mu) \quad (j = 1, \dots, r;\ l = 0, \dots, m_j - 1),$$

where the numbers $m_j$ are partial multiplicities of $T$ at $\mu$. Then the set of vectors $\{y_l^{(j)} : 1 \leq j \leq r,\ 0 \leq l \leq m_j - 1\}$ is a CSEAV of $T$ at $\mu$, and $\overline{\nu}(y_0^{(j)}) = m_j$ for $j = 1, \dots, r$.

See Proposition 1.6.3 [31, page 28].

**Definition 2.64.** [41, page 17]. Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$ such that $\rho(T) \neq \emptyset$ and $\mu \in \sigma(T)$. Let $y$ be an eigenvector belonging to $\mu$. Then $y$ is said to have multiplicity $m$ if there is a system of functions $\{y_1, \dots, y_{m-1}\}$ in $\mathscr{H}$ of length $m-1$ but no system of length $m$ associated with $y$.

**Proposition 2.65.** Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$ such that $\rho(T) \neq \emptyset$ and $\mu \in \sigma(T)$. We set $r := \operatorname{nul} T(\mu)$ and let $m_j$, $j = 1, \dots, r$, be the partial multiplicities of $T$ at $\mu$. Then there is a CSEAV $\{y_l^{(j)} : 1 \leq j \leq r,\ 0 \leq l \leq m_j - 1\}$ of $T$ at $\mu$.

See Proposition 1.6.4 [31, page 28].

**Definition 2.66.** Let $\Omega$ be a domain in $\mathbb{C}$ and $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces. Let $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$ and $\mu \in \sigma(T)$. Then $\mu$ is called a semi-simple eigenvalue of $T$ if for each $y \in N(T(\mu))\backslash\{0\}$ there is a $v \in N(T^*(\mu))$ such that

$$\langle (\frac{d}{d\lambda}T)(\mu)y, v\rangle \neq 0.$$

If $\mu \in \sigma(T)$ is semi-simple and nul $T(\mu) = 1$, then $\mu$ is called a simple eigenvalue.

See Definition 1.7.1 of [31, page 31].

**Proposition 2.67.** *Let $\Omega$ be a domain in $\mathbb{C}$ and $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces. Let $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$, $\rho(T) \neq \emptyset$ and $\mu \in \sigma(T)$. The following properties are equivalent:*

*i) $\mu$ is a semi-simple eigenvalue of $T$;*

*ii) there are CSRFs $\{y_1, \dots, y_r\}$ of $T$ at $\mu$ and $\{v_1, \dots, v_r\}$ of $T^*$ of $\mu$ such that*

$$\langle (\frac{d}{d\lambda}T)(\mu)y_i(\mu), v_j(\mu)\rangle = \delta_{ij} \quad (i, j = 1, \dots, r);$$

*iii) for each root function $y$ of $T$ at $\mu$ we have $\nu(y) = 1$;*

*iv) the pole order of $T^{-1}$ at $\mu$ is 1.*

See Proposition 1.7.2. [31, page 31].

**Proposition 2.68.** *Let $\Omega$ be a domain in $\mathbb{C}$ and $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces. Let $T \in H(\Omega, \Phi(\mathscr{H}, \mathscr{H}_1))$, $\rho(T) \neq \emptyset$ and $\mu \in \sigma(T)$. The following properties are equivalent:*

*i) $\mu$ is a semi-simple eigenvalue of $T$;*

*ii) there are bases $\{y_1, \dots, y_r\}$ of $N(T(\mu))$ and $\{v_1, \dots, v_r\}$ of $N(T^*(\mu))$ such that*

$$\langle (\frac{d}{d\lambda}T)(\mu)y_i, v_j\rangle = \delta_{ij} \quad (i, j = 1, \dots, r);$$

*iii) each eigenvalue of $T$ at $\mu$ has* rank 1*;*

*iv) the pole order of $T^{-1}$ at $\mu$ is 1.*

See Proposition 1.7.3. [31, page 32].

Definition 2.58, Definition 2.66, Proposition 2.67 and Proposition 2.68 yield the following corollary

**Corollary 2.69.** *The geometric and the algebraic multiplicities of a semi-simple eigenvalue are equal.*

**Definition 2.70.** [38, page 337] Let $\Omega$ be an open nonempty subset of $\mathbb{C}$, $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces. Let $\mathscr{L}(\mathscr{H}, \mathscr{H}_1)$ be the space of bounded linear operators from $\mathscr{H}$ to $\mathscr{H}_1$. We assume that the operator function $T : \Omega \to \mathscr{L}(\mathscr{H}, \mathscr{H}_1)$ is holomorphic. For an operator $K \in \mathscr{L}(\mathscr{H}, \mathscr{H}_1)$ we write

$$T(K)(\lambda) := T(\lambda) + K. \tag{2.15}$$

**Theorem 2.71.** *Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces. Let $K_0 \in \mathscr{L}(\mathscr{H}, \mathscr{H}_1)$ and $\lambda(K_0) \in \Omega$ such that $T(K_0)(\lambda(K_0)) \in \Phi(\mathscr{H}, \mathscr{H}_1)$ and $\lambda(K_0)$ an isolated simple eigenvalue of $T(K_0)$. Then there is a neighborhood $U$ of $K_0$ in $\mathscr{L}(\mathscr{H}, \mathscr{H}_1)$ and a closed disk $\overline{B}(\lambda_0(K_0), \varepsilon)$ in $\Omega$ with center $\lambda(K_0)$ and radius $\varepsilon > 0$ such that for each $K_1 \in U$ there is exactly one point $\lambda(K_1)$ of $\sigma(T(K_1))$ inside the open disk $B(\lambda(K_0), \varepsilon)$, and $\sigma(T(K_1)) \cap \Gamma = \emptyset$, where $\Gamma$ is the boundary of $B(\lambda(K_0), \varepsilon)$. The map $\lambda : U \to \mathbb{C}$ is continuous.*

See Theorem 2.3 of [38, page 337].

**Theorem 2.72.** *Let the hypotheses and notations of Theorem 2.71 hold. The map $\lambda : U \to \mathbb{C}$ is differentiable at $K_0$, and the derivative is given by*

$$\lambda'(K_0)K = -\langle Ku, v\rangle, \;\; K \in \mathscr{L}(\mathscr{H}, \mathscr{H}_1), \tag{2.16}$$

*where $(u, v)$ is a biorthogonal system of eigenvectors of $T(K_0)$ and $T(K_0)^*$ at $\lambda(K_0)$.*

See Theorem 2.4 of [38, page 339].

**Theorem 2.73.** *Let*

$$L(\lambda, \varepsilon) = \lambda^2 + \lambda D(\varepsilon) + K$$

*be a perturbed quadratic matrix function with perturbing parameter $\varepsilon \in \mathbb{R}$, where $D(\varepsilon)$ is an analytic hermitian matrix function and $K$ is a negative definite definite matrix on $\ker L(\lambda_0, 0)$.*

*Let $0 \neq \lambda_0 \in \mathbb{R}$ be an eigenvalue of $L(\lambda, 0)$. Then every eigenvalue of $L(\lambda, \varepsilon)$ near $\lambda_0$ is analytic.*

See Theorem 3.3 of [45, page 145].

## 2.5 Abstract boundary eigenvalue operator functions

Let $\Omega$ be a subset of $\mathbb{C}$. Let us consider the Hilbert spaces $E$, $G$, $F_1$, $F_2$, $F = F_1 \oplus F_2$ and operator functions $T \in H(\Omega, \mathscr{L}(E, F))$ and $Z \in H(\Omega, \mathscr{L}(G, E))$.

**Remark 2.74.** [31, page 46]. According to $F = F_1 \oplus F_2$, there exist $T_1 \in H(\Omega, \mathscr{L}(E, F_1))$ and $T_2 \in H(\Omega, \mathscr{L}(E, F_2))$ such that $T(\lambda) = \begin{pmatrix} T_1(\lambda) \\ T_2(\lambda) \end{pmatrix}$ for $\lambda \in \Omega$.

**Assumption 2.75.** [31, page 46]. We assume for all $\lambda \in \Omega$

$$\begin{cases} \text{i)} \quad T_1 \quad \text{is right invertible,} \\ \text{ii)} \quad Z(\lambda) \quad \text{is invertible,} \\ \text{iii)} \quad N(T_1(\lambda)) = R(Z(\lambda)). \end{cases} \tag{2.17}$$

**Remark 2.76.** [31, page 47]. Condition (2.17) means that there is an operator $U(\lambda) \in \mathscr{L}(F_1, E)$ such that $T_1(\lambda)U(\lambda) = id_{F_1}$.

**Proposition 2.77.** [31, page 47]. *We set*

$$M(\lambda) := T_2(\lambda)Z(\lambda) \qquad (\lambda \in \Omega), \tag{2.18}$$

*whence $M \in H(\Omega, \mathscr{L}(G, F_2))$.*

**Definition 2.78.** [31, page 47]. We call $T$ an abstract boundary eigenvalue operator function, $Z$ a "fundamental matrix" function and $M$ the characteristic "matrix" function associated to $T$ (with respect to $Z$).

**Theorem 2.79.** *There are operator functions*

$$C \in H(\Omega, \mathscr{L}(F_2 \oplus F_1, F)), \quad D \in H(\Omega, \mathscr{L}(E, G \oplus F_1))$$

*such that for $\lambda \in \Omega$ the operator $C(\lambda)$ and $D(\lambda)$ are invertible and the factorization*

$$T(\lambda) = C(\lambda) \begin{pmatrix} M(\lambda) & 0 \\ 0 & id_{F_1} \end{pmatrix} D(\lambda)$$

*holds.*

See Theorem 1.11.1 [31, page 47].

**Proposition 2.80.** [31, page 48] *Let $T \in H(\Omega, \mathscr{L}(E, F))$ and assume that there are Hilbert spaces $X_1$, $X_2$, $Y_1$, $Y_2$ and holomorphic functions*

$$C \in H(\Omega, \mathscr{L}(Y_1 \oplus Y_2, F)), \quad D \in H(\Omega, \mathscr{L}(E, X_1 \oplus X_2)),$$

$$M \in H(\Omega, \mathscr{L}(X_1, Y_1)), \quad J \in H(\Omega, \mathscr{L}(X_2, Y_2))$$

*such that*

$$T(\lambda) = C(\lambda) \begin{pmatrix} M(\lambda) & 0 \\ 0 & J(\lambda) \end{pmatrix} D(\lambda) \quad (\lambda \in \Omega).$$

*We suppose that the operators $C(\lambda)$, $D(\lambda)$, and $J(\lambda)$ are invertible for all $\lambda \in \Omega$.*

1. *There are operator functions*

$$C_1 \in H(\Omega, \mathscr{L}(F, Y_1)), \quad C_2 \in H(\Omega, \mathscr{L}(F, Y_2)),$$

$$D_1 \in H(\Omega, \mathscr{L}(X_1, E)), \quad D_2 \in H(\Omega, \mathscr{L}(X_2, E))$$

   *such that*

$$C^{-1}(\lambda) = \begin{pmatrix} C_1(\lambda) \\ C_2(\lambda) \end{pmatrix}, \quad D^{-1}(\lambda) = (D_1(\lambda), D_2(\lambda)) \qquad (\lambda \in \Omega).$$

2. *$T \in H(\Omega, \Phi(E, F))$ if and only if $M \in H(\Omega, \Phi(X_1, Y_1))$ and $\rho(T) = \rho(M)$*

3. $\mathrm{nul}\ T(\mu) = \mathrm{nul}\ M(\mu)$ *for all $\mu \in \sigma(T)$.*

## 2.6 The boundary eigenvalue problem

**Definition 2.81.** [31, page 3] For a set $G$ we denote the set of $k \times n$ matrices with entries from $G$ by $M_{k,n}(G)$. We write $M_n(G)$ if $n = k$.

Let $\Omega$ be a domain in $\mathbb{C}$, $-\infty < a < b < \infty$, and $n \in \mathbb{N}\backslash\{0\}$. Let $A \in H(\Omega, M_n(L_2(a,b)))$ and $T^R \in H(\Omega, \mathscr{L}((W_1^2(a,b))^n, \mathbb{C}))$. We consider boundary eigenvalue problems of the form

$$\begin{cases} y' - A(\cdot,\lambda)y = 0, \\ \quad T^R(\lambda)y = 0, \end{cases} \tag{2.19}$$

for $\lambda \in \Omega$, where a solution $y \in (W_1^2(a,b))^n$ of the differential system in (2.19) is to be understood as a weak solution, i.e., a solution in the distributional sense.

**Definition 2.82.** [31, page 102]. If we take

$$T^R(\lambda)y = W^a(\lambda)y(a) + W^b(\lambda)y(b) \tag{2.20}$$

for $y \in (W_1^2(a,b))^n$ and $\lambda \in \Omega$, where $W^a, W^b \in H(\Omega, M_n(\mathbb{C}))$, then (2.19) is two-point boundary eigenvalue problem.

**Definition 2.83.** [31, page 103]. Let

$$\begin{cases} T^D(\lambda)y := y' - A(\cdot,\lambda)y, \\ \quad T(\lambda)y := \begin{pmatrix} T^D(\lambda)y \\ T^R(\lambda)y \end{pmatrix}, \end{cases} \quad (y \in (W_1^2(a,b))^n, \lambda \in \Omega). \tag{2.21}$$

$T$ is called a boundary eigenvalue operator function.

**Proposition 2.84.** [31, page 103]. *Let $T$ as defined in Definition 2.83. Then*

$$T \in H(\Omega, \mathscr{L}((W_1^2(a,b))^n, (L_2(a,b))^n \oplus \mathbb{C}^n)).$$

**Proposition 2.85.** *Let $n \in \mathbb{N}$, $n \geq 2$. Consider the $n$-th order differential equation*

$$y^{(n)} + \sum_{i=0}^{n-1} p_i(\cdot,\lambda)y^{(i)} = 0 \quad (y \in W_n^2(a,b),\ \lambda \in \Omega), \tag{2.22}$$

*where* $p_i \in H(\Omega, L_2(a,b))$ $(i = 0, \dots, n-1)$*. Let* $L^D$ *be the differential operator*

$$L^D(\lambda)y := y^{(n)} + \sum_{i=0}^{n-1} p_i(\cdot, \lambda) y^{(i)} \quad (y \in W_n^2(a,b), \lambda \in \Omega). \tag{2.23}$$

*Then* $L^D \in H(\Omega, \mathscr{L}(W_n^2(a,b), L_2(a,b)))$*.*

See Lemma 6.1.1. [31, pages 252-253].

**Remark 2.86.** [31, page 253]. We can associate a first order system to the $n$-th order differential equation (2.22). This first order system is defined by the operator

$$T^D(\lambda)y := y' - A(\cdot, \lambda)y \quad (y \in (W_1^2(a,b))^n, \lambda \in \mathbb{C}), \tag{2.24}$$

where

$$A := (\delta_{i,j-1} - \delta_{i,n} p_{j-1})_{i,j=1}^n = \begin{pmatrix} 0 & 1 & & & & \\ & \cdot & \cdot & \cdot & & 0 \\ & \cdot & \cdot & \cdot & \cdot & \\ & 0 & \cdot & \cdot & 0 & 1 \\ -p_0 & \cdot & \cdot & \cdot & & -p_{n-1} \end{pmatrix}. \tag{2.25}$$

**Proposition 2.87.** [31, page 253]. *Let* $y \in W_n^2(a,b)$*,* $\lambda \in \Omega$*, and set*

$$\eta := \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}.$$

*Then* $\eta \in (W_1^2(a,b))^n$ *and*

$$T^D(\lambda)\eta := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ L^D(\lambda)y \end{pmatrix}.$$

**Proposition 2.88.** *Let* $\eta \in (W_1^2(a,b))^n$*,* $\lambda \in \Omega$*, and assume that* $e_i^T T^D(\lambda)\eta = 0$ *for* $i =$

$1, \ldots, n-1$. *Then* $y := e_1^\top \eta \in W_n^2(a,b)$,

$$\eta = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \tag{2.26}$$

*and*

$$L^D(\lambda)y = e_n^T T^D(\lambda)\eta. \tag{2.27}$$

See Proposition 6.1.3. [31, page 254].

**Definition 2.89.** Let $\lambda_0 \in \Omega$. A matrix function $Y_0 \in M_n(W_1^2(a,b))$ is called a fundamental matrix function of $T^D\eta = 0$ if for each $\eta \in N(T^D(\lambda_0))$ there is $c \in \mathbb{C}^n$ such that $\eta = Y_0 c$. A matrix function $Y : \Omega \to M_n(W_1^2(a,b))$ is called a fundamental matrix function of $T^D\eta = 0$ if $Y(\lambda)$ is a fundamental matrix of $T^D(\lambda)\eta = 0$ for each $\lambda \in \Omega$.

See Definition 2.5.2. [31, page 69].

**Proposition 2.90.** *Let $\lambda_0 \in \Omega$, and $Y_1$ be a fundamental matrix of $T^D\eta = 0$. Then $Y_0 \in M_n(W_1^2(a,b))$ is a fundamental matrix of $T^D(\lambda)\eta = 0$ if and only if $Y_0 = Y_1 C$ for some invertible matrix $C \in M_n(\mathbb{C})$.*

See Proposition 2.5.4. [31, page 72].

**Theorem 2.91.** *Let $T^D$ be given by (2.21). Then there is a fundamental matrix function $Y \in H(\Omega, M_n(W_1^2(a,b)))$ of $T^D\eta = 0$ with $Y(a,\lambda) = id_{\mathbb{C}^n}$ for all $\lambda \in \Omega$. In addition, $Y(\cdot,\lambda)$ is invertible in $M_n(W_1^2(a,b))$ for all $\lambda \in \Omega$.*

See Theorem 2.5.3. [31, page 69].

**Definition 2.92.** Let $\lambda_0 \in \Omega$ and $y_1, \ldots, y_n \in W_n^2(a,b)$. Then $\{y_1, \ldots, y_n\}$ is called a fundamental system of $L^D(\lambda_0)y = 0$ if for each $y \in N(L^D(\lambda_0))$ there are $c_j \in \mathbb{C}$ $(j = 1, \ldots, n)$ such that

$$y = \sum_{j=1}^n c_j y_j.$$

A function $(y_1, \ldots, y_n) : \Omega \to M_{1,n}(W_n^2(a,b))$ is called a fundamental system function of $L^D y = 0$ if $\{y_1(\lambda), \ldots, y_n(\lambda)\}$ is a fundamental system of $L^D(\lambda)y = 0$ for each $\lambda \in \Omega$.

See Definition 6.1.4. [31, page 254].

**Proposition 2.93.** *Let $\lambda_0 \in \Omega$ and $Y_0 \in M_n(W_n^2(a,b))$ be a fundamental matrix of $T^D(\lambda)\eta = 0$. Then $\{e_1^T Y_0 e_1, \dots, e_1^T Y_0 e_n\}$ is fundamental system of $L^D(\lambda_0)y = 0$, and*

$$(e_1^T Y_0 e_j)^{(i-1)} = e_1^T Y_0 e_j \tag{2.28}$$

*holds for $i = 1, \dots, n$ and $j = 1, \dots, n$.*

See Lemma 6.1.5. [31, pages 254-255].

**Proposition 2.94.** *Let $\lambda_0 \in \Omega$ and $y_1, \dots, y_n \in W_n^2(a,b)$ such that $\{y_1, \dots, y_n\}$ is a fundamental system of $L^D(\lambda_0)y = 0$. Then $(y_j^{(i-1)})_{i,j=1}^n \in M_n(W_1^2(a,b))$ is a fundamental system of $T^D(\lambda_0)\eta = 0$.*

See Lemma 6.1.6. [31, page 255].

**Proposition 2.95.** *Let $\lambda_0 \in \Omega$ and $y_1, \dots, y_n \in \Omega$. Then the following are equivalent:*

i) *$y_1, \dots, y_n$ are linearly independent, $L^D(\lambda_0)y_j = 0$ for each $j \in \{1, \dots, n\}$ and for each $y \in N(L^D(\lambda))$ there are $c_j \in \mathbb{C}$ $(j = 1, \dots, n)$ such that*

$$y = \sum_{j=1}^{n} c_j y_j;$$

ii) *$\{y_1, \dots, y_n\}$ is a fundamental system of $L^D(\lambda_0)y = 0$;*

iii) *$(y_j^{(i-1)})_{i,j=1}^n \in M_n(W_1^2(a,b))$ is a fundamental system of $T^D(\lambda_0)\eta = 0$.*

See Proposition 6.1.7. [31, page 255].

**Theorem 2.96.** *There is a fundamental system $(y_1, \dots, y_n)$ of $L^D y = 0$ such that $y_j^{(i-1)}(a, \lambda) = \delta_{i,j}$ for $\lambda \in \Omega$ and $i, j = 1, \dots, n$. Furthermore, the fundamental system function is uniquely determined and depends holomorphically on $\lambda \in \Omega$. More precisely, we have $y_j \in H(\Omega, W_n^2(a,b))$ for $j = 1, \dots, n$.*

See Theorem 6.1.8 [31, page 256].

**Proposition 2.97.** [31, pages 256-257]. *Let* $L^R \in H(\Omega, \mathscr{L}(W_n^2(a,b), \mathbb{C}^n))$. *Then for each* $\lambda \in \Omega$, *there is an operator* $T^R(\lambda) \in \mathscr{L}((W_1^2(a,b))^n, \mathbb{C}^n)$ *such that*

$$L^R(\lambda)y = T^R(\lambda) \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix} \tag{2.29}$$

*for all* $y \in W_n^2(a,b)$.

**Proposition 2.98.** [31, page 257]. *Let*

$$L^R(\lambda)y := \left( \sum_{j=1}^{n} (\alpha_{ij}(\lambda) y^{(j-1)}(a) + \beta_{ij}(\lambda) y^{(j-1)}(b)) \right)_{i=1}^{n} \qquad (y \in W_n^2(a,b)),$$

*where* $\alpha_{ij}$ *and* $\beta_{ij}$ *are complex valued functions and where* $L^R$ *depends holomorphically on* $\lambda$. *Then* $\alpha_{ij}$ *and* $\beta_{ij}$ *are holomorphic functions and* $T^R(\lambda)$, *defined by*

$$T^R(\lambda)y := (\alpha_{ij}(\lambda))_{i,j=1}^{n} y(a) + (\beta_{ij}(\lambda))_{i,j=1}^{n} y(b) \quad (y \in W_n^2(a,b)),$$

*depends holomorphically on* $\lambda$.

**Definition 2.99.** [31, page 259]. Let $L^D$ and $L^R$ be as respectively defined in Proposition 2.85 and Proposition 2.98. We call

$$L = (L^D, L^R) \in H(\Omega, \mathscr{L}(W_n^2(a,b), L_2(a,b) \oplus \mathbb{C}^n)) \tag{2.30}$$

a boundary eigenvalue operator function.

**Definition 2.100.** [31, page 259]. Let $L$ be the operator function defined in (2.30) and $\{y_1, \dots, y_n\}$ be the fundamental system function of $L^D y = 0$, given by Theorem 2.96. Set $Y := (y_j^{(i-1)})_{i,j=1}^{n}$. Define

$$Z_L(\lambda)c := (y_1(\cdot, \lambda), \dots, y_n(\cdot, \lambda))c = e_1^T Y(\cdot, \lambda)c \quad (c \in \mathbb{C}^n, \lambda \in \Omega). \tag{2.31}$$

Then a characteristic matrix of the boundary eigenvalue operator function $L$ is defined by

$$M(\lambda) = L^R(\lambda) Z_L(\lambda) \tag{2.32}$$

and

$$\Delta(\lambda) = \det M(\lambda) \tag{2.33}$$

is called the characteristic determinant of the boundary eigenvalue operator function $L$.

**Remark 2.101.** [31, page 260]. Note that the matrix $M$ defined in Proposition 2.100 is also a characteristic matrix function of the associated first order boundary eigenvalue operator function $T = (T^D, T^R)$ given by (2.24) and (2.29).

**Theorem 2.102.** *The operator function $L$, defined in* (2.30)*, is an abstract boundary eigenvalue operator function in the sense of Section 2.5.*

See Theorem 6.3.1. [31, page 260].

**Proposition 2.103.** [41, pages 14-15] *Let $\lambda \in \mathbb{C}$ and $T$ be a linear differential operator defined in a Hilbert space $\mathscr{H}$, such that $Ty = \lambda y$ and $\Delta(\lambda)$ be the characteristic determinant of $T$. Then*

1. *$\Delta(\lambda)$ is an analytic function of $\lambda$.*
2. *The eigenvalues of the operator $T$ are the zeros of the function $\Delta(\lambda)$.*
3. *If $\Delta(\lambda)$ vanishes identically, then any number $\lambda$ is an eigenvalue of the operator $T$.*
4. *If, however, $\Delta(\lambda)$ is not identically zero, then the operator $T$ has at most countably many eigenvalues, and these eigenvalues can have no finite limit-point.*

**Theorem 2.104.** *Let $\{y_1, \ldots, y_n\}$ be the fundamental system function of $L^D y = 0$, given by Theorem 2.96. Set $Y := (y_j^{(i-1)})_{i,j=1}^n$. Define*

$$(U_L(\lambda)f)(x) := e_1^T Y(x,\lambda) \int_0^x Y(t,\lambda)^{-1} e_n f(t) dt \quad (f \in L_2(a,b)). \tag{2.34}$$

*Then the eigenvalue operator function $L$ given by* (2.30) *is holomorphically equivalent on $\Omega$ to the $L_2(a,b)$-extension of $M$; more precisely, for $\lambda \in \Omega$ we have*

$$L(\lambda) = \begin{pmatrix} 0 & id_{L_2(a,b)} \\ id_{\mathbb{C}^n} & L^R(\lambda)U_L(\lambda) \end{pmatrix} \begin{pmatrix} M(\lambda) & 0 \\ 0 & id_{L_2(a,b)} \end{pmatrix} (Z_L(\lambda), U_L(\lambda))^{-1},$$

*and the operator*

$$\begin{pmatrix} 0 & id_{L_2(a,b)} \\ id_{\mathbb{C}^n} & L^R(\lambda)U_L(\lambda) \end{pmatrix} \in \mathscr{L}(\mathbb{C}^n \oplus L_2(a,b), L_2(a,b) \oplus \mathbb{C}^n)$$

*and*

$$(Z_L(\lambda), U_L(\lambda)) \in \mathscr{L}(\mathbb{C}^n \oplus L_2(a,b), W_n^2(a,b))$$

*are invertible and depend holomorphically on* $\lambda$*.*

See Theorem 6.3.2. of [31, page 261].

**Corollary 2.105.** *The eigenvalue operator function* $L(\cdot,\alpha)$ *is a Fredholm operator valued function and* $\rho(L) = \rho(M) = \rho(T)$*.*

See Corollary 6.3.3. of [31, page 261].

**Definition 2.106.** [31, page 282]. Let $-\infty < a < b < \infty$, $n \in \mathbb{N}$, $n \geq 2$. Let

$$p_i(\cdot,\lambda) = \sum_{j=0}^{n-i} \lambda^j \pi_{n-i,j} \qquad (i = 0,\dots,n-1), \tag{2.35}$$

where $\pi_{n-i,j} \in L_2(a,b)$ $(i = 0,\dots,n-1, j = 0,\dots,n-1)$. We assume $\pi_{n-i,n-i} \neq 0$ for some $i \in \{0,\dots,n-1\}$. The function $\pi$ defined by

$$\pi(\cdot,\rho) := \rho^n + \sum_{i=0}^{n-1} \rho^i \pi_{n-i,n-i} \qquad (\rho \in \mathbb{C}) \tag{2.36}$$

is called the characteristic function of the differential equation (2.23).

**Proposition 2.107.** *Together with the assumptions of Definition* (2.106)*, let* $w_{ki}$ $(i,k = 1,\dots,n)$ *be polynomials in* $\lambda$ *with coefficients in* $L_1(a,b)$ *and* $w_{ki}^{(j)}$ $(j = 0,1)$*,* $i,k = 1,\dots,n)$ *be polynomials in* $\lambda$ *with complex coefficients. We consider the boundary eigenvalue problem*

$$L^D(\lambda)y = 0 \text{ and } L^R(\lambda)y = 0, \tag{2.37}$$

*where*

$$L^D(\lambda)y := y^{(n)} + \sum_{i=0}^{n-1} p_i(\cdot,\lambda) y^{(i)}, \tag{2.38}$$

$$L^R(\lambda)y := \left( \sum_{i=1}^{n} \left( w_{ki}^{(0)} y^{(i-1)}(a) + w_{ki}^{(1)} y^{(i-1)}(b) \right) + \sum_{i=1}^{n} \int_a^b w_{ki}(\xi,\lambda) y^{(i-1)}(\xi) d\xi \right)_{k=1}^{n}, \tag{2.39}$$

*where* $\lambda \in \mathbb{C}$ *and* $y \in W_n^2(a,b)$*. Then*

$$L := (L^D, L^R) \in H(\mathbb{C}, \mathscr{L}(W_n^2(a,b), L_2(a,b) \oplus \mathbb{C}^n)). \tag{2.40}$$

See Proposition 7.1.1 of [31, pages 281].

# Chapter 3

# Spectral Properties of Self-adjoint Fourth Order Differential Operators

## 3.1 Introduction

On the interval $[0, a]$ we consider the eigenvalue problem

$$y^{(4)} - (gy')' = \lambda^2 y, \tag{3.1}$$

$$B_j(\lambda)y = 0, \ j = 1, 2, 3, 4, \tag{3.2}$$

where $a > 0$, $g \in C^1[0, a]$ is a real valued function and (3.2) are separated boundary conditions where the $B_j(\lambda)$ are constant or depend on $\lambda$ linearly. We recall that the quasi-derivatives associated with (3.1) are given by

$$y^{[0]} = y, \ y^{[1]} = y', \ y^{[2]} = y'', \ y^{[3]} = y^{(3)} - gy', \ y^{[4]} = y^{(4)} - (gy')',$$

see Definition 2.3. The boundary conditions (3.2) are taken at the endpoint 0 for $j = 1, 2$ and at the endpoint $a$ for $j = 3, 4$. Further, we assume for simplicity that either $B_j(\lambda)y = y^{[p_j]}(a_j) + i\varepsilon_j \alpha\lambda y^{[q_j]}(a_j)$ or $B_j(\lambda)y = y^{[p_j]}(a_j)$, where $a_j = 0$ for $j = 1, 2$, $a_j = a$ for $j = 3, 4$, $p_j + q_j = 3$ for $j = 1, 2, 3, 4$, $\alpha > 0$ and $\varepsilon_j \in \{-1, 1\}$.

We define

$$\Theta_1 = \{s \in \{1,2,3,4\} : B_s(\lambda) \text{ depends on } \lambda\}, \ \Theta_0 = \{1,2,3,4\} \backslash \Theta_1,$$

$$\Theta_1^0 = \Theta_1 \cap \{1,2\}, \quad \Theta_1^a = \Theta_1 \cap \{3,4\}$$

and put

$$k = |\Theta_1|. \tag{3.3}$$

**Assumption 3.1.** We assume that the numbers $p_1$, $p_2$, $q_j$ for $j \in \Theta_1^0$ are distinct and that the numbers $p_3$, $p_4$, $q_j$ for $j \in \Theta_1^a$ are distinct.

This means that for any pair $(r, a_j)$ the term $y^{[r]}(a_j)$ occurs at most once in the boundary conditions (3.2).

We formally denote the collection of the four boundary conditions (3.2) by $U$ and define the following operators related to $U$:

$$U_0 y = (y^{[p_j]}(a_j))_{j \in \Theta_1} \text{ and } U_1 y = (\varepsilon_j y^{[q_j]}(a_j))_{j \in \Theta_1}, \quad y \in W_4^2(0,a). \tag{3.4}$$

Let $l = 4 - k$ and define

$$\Theta_0 =: \{\sigma_1, \dots, \sigma_l\}, \qquad \Theta_1 =: \{\theta_1, \dots, \theta_k\}, \tag{3.5}$$

$$\widetilde{p}_i := \begin{cases} p_i + 1 & \text{if } i = 1,2, \\ p_i + 5 & \text{if } i = 3,4, \end{cases} \qquad \widetilde{q}_i := \begin{cases} q_i + 1 & \text{if } i = 1,2, \\ q_i + 5 & \text{if } i = 3,4, \end{cases} \tag{3.6}$$

$$V := (\delta_{\widetilde{p}_{\sigma_i},j})_{i=1,\dots,l;\ j=1,\dots,8}, \tag{3.7}$$

$$V_0 = \left(\delta_{\widetilde{p}_{\theta_i},j}\right)_{i=1,\dots,k;\ j=1,\dots,8} \quad \text{and} \quad V_1 = \left(\varepsilon_{\theta_i}\delta_{\widetilde{q}_{\theta_i},j}\right)_{i=1,\dots,k;\ j=1,\dots,8}. \tag{3.8}$$

Then the operators $U_0$ and $U_1$ defined in (3.4) can be written as

$$U_0 y = V_0 Y_R \quad \text{and} \quad U_1 y = V_1 Y_R, \tag{3.9}$$

where $Y_R = \begin{pmatrix} Y(0) \\ Y(a) \end{pmatrix}$ with $Y = \left(y^{[0]}, y^{[1]}, y^{[2]}, y^{[3]}\right)^\top$, see [37, (1.6)].

Note that we have defined the sets $\Theta_0$ and $\Theta_1$ in (3.5), the numbers $\widetilde{p}_i$ and $\widetilde{q}_i$ in (3.6) and have used these definitions to define the matrices $V$, $V_0$ and $V_1$. We have used these matrices to represent the infinite dimensional problem (3.1)–(3.2) by finite dimensional matrix problem, see (3.7)–(3.8) and (3.11)–(3.13).

We will associate a quadratic operator pencil

$$L(\lambda,\alpha)=\lambda^2 M-i\alpha\lambda K-A(U),\quad \lambda\in\mathbb{C} \tag{3.10}$$

in the space $L_2(0,a)\oplus\mathbb{C}^k$ with this problem, where

$$K=\begin{pmatrix}0&0\\0&I\end{pmatrix}\text{ and }M=\begin{pmatrix}I&0\\0&0\end{pmatrix}$$

are bounded self-adjoint operators.

The maximal differential operator $A_{\max}$ associated with the boundary value problem (3.1) and those boundary conditions from (3.2) which depend on $\lambda$ is defined on $L_2(0,a)\oplus\mathbb{C}^k$ by

$$\mathscr{D}(A_{\max})=\left\{\widetilde{y}=\begin{pmatrix}y\\U_1y\end{pmatrix},\, y\in W_4^2(0,a)\right\},\quad A_{\max}\widetilde{y}=\begin{pmatrix}y^{[4]}\\U_0y\end{pmatrix},\quad \widetilde{y}\in\mathscr{D}(A_{\max}). \tag{3.11}$$

Now we define the operator $A=A(U)$ on $L_2(0,a)\oplus\mathbb{C}^k$ such that

$$\mathscr{D}(A(U))=\left\{\widetilde{y}=\begin{pmatrix}y\\U_1y\end{pmatrix},\ y\in W_4^2(0,a),\ y^{[p_j]}(a_j)=0\text{ for }j\in\Theta_0\right\} \tag{3.12}$$

$$(A(U))\widetilde{y}=\begin{pmatrix}y^{[4]}\\U_0y\end{pmatrix}\ \text{ for }\widetilde{y}\in\mathscr{D}(A(U)). \tag{3.13}$$

This means that $A(U)$ is the restriction of $A_{\max}$ with respect to the boundary conditions $VY_R=0$ where $Y_R=\begin{pmatrix}Y(0)\\Y(a)\end{pmatrix}$ with $Y=\begin{pmatrix}y^{[0]}, & y^{[1]}, & y^{[2]}, & y^{[3]}\end{pmatrix}^\top$, see [51, page 26].

We investigate in this chapter the resolvent set of the operator pencil $L(\cdot,\alpha)$ and its spectrum and in the following chapter the asymptotics of the eigenvalues of the corresponding boundary problems for which the main operator $A(U)$ is self-adjoint. The conditions for which the differential operator $A(U)$ is self-adjoint are given in the following theorem which is Theorem 4.5 of [39].

**Theorem 3.2.** *Denote by $P_0$ the set of $p$ in $y^{[p]}(0) = 0$ for the $\lambda$-independent boundary conditions and by $P_a$ the corresponding set for $y^{[p]}(a) = 0$. Then the differential operator $A(U)$ associated with this boundary value problem is self-adjoint if and only if $p + q = 3$ for all boundary conditions of the form $y^{[p]}(a_j) + i\alpha\varepsilon_j\lambda y^{[q]}(a_j) = 0$ and $\varepsilon_j = 1$ if $q$ is even in case $a_j = 0$ or odd in case $a_j = a$, $\varepsilon_j = -1$ otherwise, $\{0,3\} \not\subset P_0$, $\{1,2\} \not\subset P_0$, $\{0,3\} \not\subset P_a$ and $\{1,2\} \not\subset P_a$.*

## 3.2 The resolvent set of the operator pencil $L(\cdot, \alpha)$

**Proposition 3.3.** *$M + K = I$ and $M|_{\mathscr{D}(A(U))} > 0$.*

*Proof.*

$$(M+K) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Hence $M+K = I_{L_2(0,a)\oplus\mathbb{C}^k}$. Let $\widetilde{y} \in \mathscr{D}(A(U))$ such that $\widetilde{y} \neq 0$. We know that $\widetilde{y} = \begin{pmatrix} y \\ U_1 y \end{pmatrix}$, so if $y = 0$, then $\widetilde{y} = 0$. Thus if $\widetilde{y} \neq 0$, then $y \neq 0$. Whence $(M\widetilde{y}, \widetilde{y}) = \widetilde{y}^* M \widetilde{y} = (\bar{y}, \overline{U_1 y}) \begin{pmatrix} y \\ 0 \end{pmatrix} = |y|^2 > 0$. Therefore $M|_{\mathscr{D}(A(U))} > 0$. ☐

**Definition 3.4.** Let $Z_1 = \{y \in W_4^2(0,a) : VY_R = 0\}$. Observe that $Z_1$ is a closed finite codimensional subspace of $W_4^2(0,a)$. Let $Z_2$ be a complementary space of $Z_1$ in $W_4^2(0,a)$, let

$$\begin{aligned} p_U : W_4^2(0,a) &\longrightarrow Z_1 \\ y &\mapsto y_U \end{aligned} \tag{3.14}$$

be the projection from $W_4^2(0,a)$ onto $Z_1$ along $Z_2$. Note that $p_U$ is a bounded operator.

Let

$$\begin{aligned} s : W_4^2(0,a) &\longrightarrow L_2(0,a) \oplus \mathbb{C}^k \\ y &\mapsto \widetilde{y} = \begin{pmatrix} y \\ U_1 y \end{pmatrix} = \begin{pmatrix} y \\ V_1 Y_R \end{pmatrix}, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} r : W_4^2(0,a) &\longrightarrow \mathbb{C}^8 \\ y &\mapsto r(y) = Y_R. \end{aligned} \tag{3.16}$$

**Proposition 3.5.** *The linear map $r$ is surjective and bounded.*

*Proof.* For all polynomials $y$ we have $\widetilde{y} \in \mathscr{D}(A_{\max})$, so

$$\{Y_R : \widetilde{y} \in \mathscr{D}(A_{\max})\} = \mathbb{C}^8. \tag{3.17}$$

Thus it follows from (3.11) that the linear map $r$ is surjective. We will now prove that $r$ is bounded. For all $y \in W_4^2(0,a)$, $|r(y)| = |Y_R| < \infty$, hence $r$ is bounded. □

**Remark 3.6.** It follows from Assumption 3.1, equation (3.7) and Definition 3.4 that rank $V = 4-k$, thus $\dim Z_1 = \dim(N(V)) = 8-(4-k) = 4+k$. Since $Z_2$ is the complementary space of $Z_1$ in $W_4^2(0,a)$, then $\dim Z_2 = 8-(4+k) = 4-k = l$.

**Definition 3.7.** Choose and fix a bijective linear map

$$D_2 : \mathbb{C}^l \longrightarrow Z_2. \tag{3.18}$$

Finally let

$$D_1 : W_4^2(0,a) \longrightarrow \mathbb{C}^l \tag{3.19}$$

be defined by $D_1 = D_2^{-1}(I - p_U)$.

Define the boundary eigenvalue operator function

$$T(\lambda) : W_4^2(0,a) \longrightarrow L_2(0,a) \oplus \mathbb{C}^k \oplus \mathbb{C}^l$$

by

$$T(\lambda)y = \begin{pmatrix} y^{[4]} - \lambda^2 y \\ i\alpha\lambda U_1 y + U_0 y \\ VY_R \end{pmatrix},$$

where $\lambda \in \mathbb{C}$.

**Proposition 3.8.** *The linear map*

$$VrD_2 : \mathbb{C}^l \longrightarrow \mathbb{C}^l$$

*is invertible.*

*Proof.* The definitions of the boundary conditions (3.2), the matrix $V$, see (3.7), and Assumption 3.1 imply that $\dim R(V) = l$. As $\dim \mathbb{C}^l < \dim \mathbb{C}^8$ and $\dim R(V) = l$, then the linear map

$$V : \mathbb{C}^8 \longrightarrow \mathbb{C}^l$$

is surjective. Since the linear map

$$r : W_4^2(0, a) \longrightarrow \mathbb{C}^8$$

is surjective, the linear map

$$Vr : W_4^2(0, a) \longrightarrow \mathbb{C}^l$$

is surjective. Let $y \in Z_2$, such that $Vry = 0$. Then $VY_R = 0$. It follows from the definition of $Z_2$, see Definition 3.4, that $y = 0$. Thus the restriction of $Vr$ to $Z_2$ is injective. The linear map

$$Vr : W_4^2(0, a) \longrightarrow \mathbb{C}^l$$

is surjective and its restriction to $Z_2$ is injective. Since the restriction of the linear map $Vr$ to $Z_2$ is injective and surjective, then it is bijective. As the linear map

$$D_2 : \mathbb{C}^l \longrightarrow Z_2$$

is bijective, see Definition 3.7, the linear map

$$VrD_2 : \mathbb{C}^l \longrightarrow \mathbb{C}^l$$

is bijective thus it is invertible. □

**Proposition 3.9.** *Let $\lambda \in \mathbb{C}$. Then the operator*

$$\begin{pmatrix} -I & E_{12} \\ 0 & VrD_2 \end{pmatrix} : (L_2(0, a) \oplus \mathbb{C}^k) \oplus \mathbb{C}^l \longrightarrow (L_2(0, a) \oplus \mathbb{C}^k) \oplus \mathbb{C}^l$$

*is bijective and bounded, where* $E_{12} = \begin{pmatrix} A_0 D_2 - \lambda^2 D_2 \\ U_0 D_2 + i\alpha\lambda U_1 D_2 \end{pmatrix}$ *and for* $y \in W_4^2(0, a)$, $A_0 y = y^{[4]}$.

*Proof.* The space $Z_2$ is a subspace of $W_4^2(0,a)$ and $\dim Z_2 = l$ (see Definition 3.4 and Remark 3.6), then the linear map

$$D_2 : \mathbb{C}^l \longrightarrow Z_2$$

is bounded as well as the linear map

$$A_0 D_2 - \lambda^2 D_2 : \mathbb{C}^l \longrightarrow L_2(0,a).$$

Since $y \in W_4^2(0,a)$, then $|Y_R| < \infty$. It follows from the definitions of $U_0$ and $U_1$, see (3.4), and from the definition of $A(U)$, see (3.11)–(3.13), that $U_0 : W_4^2(0,a) \longrightarrow \mathbb{C}^k$, $y \mapsto U_0 y = V_0 Y_R$ and $U_1 : W_4^2(0,a) \longrightarrow \mathbb{C}^k$, $y \mapsto V_1 Y_R$. The matrices $V_0$ and $V_1$ are finite dimensional matrices, with entries 0,1 for $V_0$ and -1, 0, 1 for $V_1$. Then it follows from (3.9) that for $y \in W_4^2(0,a)$ $|U_0 y| = |V_0 Y_R| < \infty$ and $|U_1 y| = |V_1 Y_R| < \infty$. On the other hand, since $\dim \mathbb{C}^k = k$, then the linear operators $U_0$ and $U_1$ are bounded, and the linear map

$$U_0 D_2 + i\alpha\lambda U_1 D_2 : \mathbb{C}^l \longrightarrow \mathbb{C}^k$$

is bounded. Thus the linear map

$$E_{12} = \begin{pmatrix} A_0 D_2 - \lambda^2 D_2 \\ U_0 D_2 + i\alpha\lambda U_1 D_2 \end{pmatrix} : \mathbb{C}^l \longrightarrow L_2(0,a) \oplus \mathbb{C}^k$$

is bounded. For all $x \in \mathbb{C}^l$, $|x| < \infty$, since the linear map $V$ is a finite dimensional matrix (see (3.7)), then the linear map $V : \mathbb{C}^8 \longrightarrow \mathbb{C}^l$ is bounded. As $D_2$ and $r$ are bounded, see Proposition 3.5, the operator

$$VrD_2 : \mathbb{C}^l \longrightarrow \mathbb{C}^l$$

is bounded. The linear maps

$$I : L_2(0,a) \oplus \mathbb{C}^k \longrightarrow L_2(0,a) \oplus \mathbb{C}^k,$$

$$E_{12} : \mathbb{C}^k \longrightarrow L_2(0,a) \oplus \mathbb{C}^k$$

and

$$VrD_2 : \mathbb{C}^l \longrightarrow \mathbb{C}^l$$

are bounded, hence the operator

$$\begin{pmatrix} -I & E_{12} \\ 0 & VrD_2 \end{pmatrix} : (L_2(0,a) \oplus \mathbb{C}^k) \oplus \mathbb{C}^l \longrightarrow (L_2(0,a) \oplus \mathbb{C}^k) \oplus \mathbb{C}^l$$

is bounded. It follows from Proposition 3.8 that

$$\begin{aligned}
\begin{pmatrix} -I & E_{12}(VrD_2)^{-1} \\ 0 & (VrD_2)^{-1} \end{pmatrix}\begin{pmatrix} -I & E_{12} \\ 0 & VrD_2 \end{pmatrix} &= \begin{pmatrix} I & -E_{12}+E_{12}(VrD_2)^{-1}VrD_2 \\ 0 & (VrD_2)^{-1}VrD_2 \end{pmatrix} \\
&= \begin{pmatrix} I & -E_{12}+E_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\begin{pmatrix} -I & E_{12} \\ 0 & VrD_2 \end{pmatrix}\begin{pmatrix} -I & E_{12}(VrD_2)^{-1} \\ 0 & (VrD_2)^{-1} \end{pmatrix} &= \begin{pmatrix} I & -E_{12}(VrD_2)^{-1}+E_{12}(VrD_2)^{-1} \\ 0 & VrD_2(VrD_2)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},
\end{aligned}$$

thus the operator

$$\begin{pmatrix} -I & E_{12} \\ 0 & VrD_2 \end{pmatrix} : (L_2(0,a)\oplus\mathbb{C}^k)\oplus\mathbb{C}^l \longrightarrow (L_2(0,a)\oplus\mathbb{C}^k)\oplus\mathbb{C}^l$$

is bijective. □

**Proposition 3.10.** *Let $\lambda\in\mathbb{C}$ and $\mathscr{D}(L(\lambda,\alpha)) := \mathscr{D}(A(U))$. Then the operator*

$$\begin{pmatrix} sp_U \\ D_2^{-1}(I-p_U) \end{pmatrix} : W_4^2(0,a) \longrightarrow (L_2(0,a)\oplus\mathbb{C}^k)\oplus\mathbb{C}^l$$

*is injective and bounded with range $\mathscr{D}(L(\lambda,\alpha))\oplus\mathbb{C}^l$.*

*Proof.* It is obvious that the linear map

$$s : W_4^2(0,a) \longrightarrow L_2(0,a)\oplus\mathbb{C}^k$$

is bounded. Since $Z_2$ is a finite dimensional subspace of $W_4^2(0,a)$, then the linear map $D_2 : \mathbb{C}^l \longrightarrow Z_2$ is bounded. As $D_2 : \mathbb{C}^l \longrightarrow Z_2$ is bijective and $\dim C^l = l$, then the linear map $D_2^{-1} : Z_2 \longrightarrow \mathbb{C}^l$ is bounded. Since $p_U$, $s$ and $D_2^{-1}$ are bounded, the operator

$$\begin{pmatrix} sp_U \\ D_2^{-1}(I-p_U) \end{pmatrix} : W_4^2(0,a) \longrightarrow (L_2(0,a)\oplus\mathbb{C}^k)\oplus\mathbb{C}^l$$

is bounded.

Let $y \in W_4^2(0,a)$ such that $\begin{pmatrix} sp_U \\ D_2^{-1}(I-p_U) \end{pmatrix} y = 0$. Then $sp_U y = \begin{pmatrix} y_U \\ U_1 y_U \end{pmatrix} = 0$ and $D_2^{-1}(I - p_U)y = 0$. Since $\begin{pmatrix} y_U \\ U_1 y_U \end{pmatrix} = 0$, $y_U = 0$. As $D_2^{-1}(I-p_U)y = 0$, then $(I-p_U)y = 0$ and $y = Iy = p_U y$. Since $p_U y = y_U = 0$, then $y = p_U y = 0$. Hence the operator $\begin{pmatrix} sp_U \\ D_2^{-1}(I-p_U) \end{pmatrix}$ is injective.

Let us now prove that $\mathscr{D}(L(\lambda,\alpha)) \oplus \mathbb{C}^l$ is the range of the operator $\begin{pmatrix} sp_U \\ D_2^{-1}(I-p_U) \end{pmatrix}$.

Define $\mathcal{T} := \begin{pmatrix} sp_U \\ D_2^{-1}(I-p_U) \end{pmatrix}$. By the definitions of the operators $p_U$, $s$ and $D_2^{-1}(I-p_U)$, for all $y \in W_4^2(0,a)$, $p_U y \in Z_1$, $U_1 p_U y \in \mathbb{C}^k$ and $D_2^{-1}(I-p_U)y \in \mathbb{C}^l$. But from (3.12) and Definition 3.4, for all $y \in W_4^2(0,a)$, $sp_U y \in \mathscr{D}(A(U))$ and $D_2^{-1}(I-p_U)y \in \mathbb{C}^l$. Thus for all $y \in W_4^2(0,a)$, $\mathcal{T}y \in \mathscr{D}(A(U)) \oplus \mathbb{C}^l$. Therefore

$$R(\mathcal{T}) \subset \mathscr{D}(A(U)) \oplus \mathbb{C}^l = \mathscr{D}(L(\lambda,\alpha)) \oplus \mathbb{C}^l. \tag{3.20}$$

Conversely, let $Z \in \mathscr{D}(L(\lambda,\alpha)) \oplus \mathbb{C}^l$. Then $Z = \begin{pmatrix} \widetilde{z} \\ c \end{pmatrix}$ where $\widetilde{z} \in \mathscr{D}(L(\lambda,\alpha))$ and $c \in \mathbb{C}^l$. As $\widetilde{z} \in \mathscr{D}(L(\lambda,\alpha)) = \mathscr{D}(A(U))$, then there exists $z \in W_4^2(0,a)$ such that $\widetilde{z} = sz = \begin{pmatrix} z \\ U_1 z \end{pmatrix}$ and $VZ_R = 0$. It follows from Definition 3.4 that since $\widetilde{z} = \begin{pmatrix} z \\ U_1 z \end{pmatrix}$ and $VZ_R = 0$, then $z \in Z_1$. However $Z_1$ is a closed subspace of $W_4^2(0,a)$, while $Z_2$ is a finite dimensional complementary space of $Z_1$ in $W_4^2(0,a)$, see Definition 3.4, thus $W_4^2(0,a) = Z_1 \oplus Z_2$. We set $y = z + D_2 c$, then $y \in Z_1 \oplus Z_2 = W_4^2(0,a)$. Since $z \in Z_1$ and $D_2 c \in Z_2$, then $p_U y = p_U z = z$. Hence $sz = sp_U y$ and $(I-p_U)y = y - z = D_2 c$, it follows that $c = D_2^{-1}(I-p_U)y$. Therefore for $y = z + D_2 c$,

$$Z = \begin{pmatrix} \widetilde{z} \\ c \end{pmatrix} = \begin{pmatrix} sz \\ c \end{pmatrix} = \begin{pmatrix} sp_U y \\ D_2^{-1}(I-p_U)y \end{pmatrix} \in R(\mathcal{T}).$$

Whence

$$\mathscr{D}(L(\lambda,\alpha)) \oplus \mathbb{C}^l \subset R(\mathcal{T}). \tag{3.21}$$

It follows from (3.20) and (3.21) that $\mathscr{D}(L(\lambda,\alpha)) \oplus \mathbb{C}^l = R(\mathcal{T})$. $\square$

**Proposition 3.11.** *Let* $\lambda \in \mathbb{C}$. *Then* $T(\lambda) = \begin{pmatrix} -I & E_{12} \\ 0 & VrD_2 \end{pmatrix} \begin{pmatrix} L(\lambda,\alpha) & 0 \\ 0 & I_{\mathbb{C}^l} \end{pmatrix} \begin{pmatrix} sp_U \\ D_2^{-1}(I-p_U) \end{pmatrix}$.

*Proof.* Let $S(\lambda) := \begin{pmatrix} -I & E_{12} \\ 0 & VrD_2 \end{pmatrix} \begin{pmatrix} L(\lambda,\alpha) & 0 \\ 0 & I_{\mathbb{C}^l} \end{pmatrix} \begin{pmatrix} sp_U \\ D_2^{-1}(I-p_U) \end{pmatrix}$. Then for all $y \in W_4^2(0,a)$,

$$\begin{aligned} S(\lambda)y &= \begin{pmatrix} -I & E_{12} \\ 0 & VrD_2 \end{pmatrix} \begin{pmatrix} L(\lambda,\alpha) & 0 \\ 0 & I_{\mathbb{C}^l} \end{pmatrix} \begin{pmatrix} \widetilde{y}_U \\ D_1 y \end{pmatrix} \\ &= \begin{pmatrix} -I & E_{12} \\ 0 & VrD_2 \end{pmatrix} \begin{pmatrix} L(\lambda,\alpha)\widetilde{y}_U \\ D_1 y \end{pmatrix} \\ &= \begin{pmatrix} -L(\lambda,\alpha)\widetilde{y}_U + E_{12}D_1 y \\ VrD_2D_1 y \end{pmatrix}. \end{aligned}$$

We recall that for $y \in W_4^2(0,a)$, $\widetilde{y} = \begin{pmatrix} y \\ U_1 y \end{pmatrix}$ (see (3.11)). So for $y \in W_4^2(0,a)$, $\widetilde{y}_U = \begin{pmatrix} y_U \\ U_1 y_U \end{pmatrix}$ and

$$\begin{aligned} -L(\lambda,\alpha)\widetilde{y}_U &= -\lambda^2 M\widetilde{y}_U + i\alpha\lambda K\widetilde{y}_U + A(U)\widetilde{y}_U \\ &= \begin{pmatrix} -\lambda^2 y_U + y_U^{[4]} \\ i\alpha\lambda U_1 y_U + U_0 y_U \end{pmatrix}. \end{aligned}$$

But for all $y \in W_4^2(0,a)$,

$$\begin{aligned} E_{12}D_1 y &= \begin{pmatrix} A_0D_2 - \lambda^2 D_2 \\ U_0D_2 + i\alpha\lambda U_1 D_2 \end{pmatrix} D_1 y \\ &= \begin{pmatrix} A_0D_2D_1 y - \lambda^2 D_2D_1 y \\ U_0D_2D_1 y + i\alpha\lambda U_1 D_2D_1 y \end{pmatrix} \\ &= \begin{pmatrix} A_0(I-p_U)y - \lambda^2(I-p_U)y \\ U_0(I-p_U)y + i\alpha\lambda U_1(I-p_U)y \end{pmatrix}. \end{aligned}$$

Then for all $y \in W_4^2(0,a)$,

$$-L(\lambda,\alpha)\widetilde{y}_U + E_{12}D_1 y = \begin{pmatrix} y^{[4]} - \lambda^2 y \\ i\alpha\lambda U_1 y + U_0 y \end{pmatrix}.$$

Also for all $y \in W_4^2(0,a)$, since $D_1 = D_2^{-1}(I - p_U)$, see (3.19), then

$$VrD_2D_1y = Vr(I - p_U)y = Vry - Vrp_Uy = VY_R - Vry_U.$$

As for all $y \in W_4^2(0,a)$ $y_U \in Z_1$, then $Vry_U = 0$ and $Vr(I - p_U)y = VY_R$. Therefore for all $y \in W_4^2(0,a)$,

$$S(\lambda)y = \begin{pmatrix} y^{[4]} - \lambda^2 y \\ i\alpha\lambda U_1 y + U_0 y \\ VY_R \end{pmatrix} = T(\lambda)y.$$

□

**Corollary 3.12.** *The boundary eigenvalue operator function $T(\lambda)$ and the operator pencil $L(\cdot,\alpha)$ have the same resolvent set.*

*Proof.* For all $\lambda \in \mathbb{C}$, the operator $\begin{pmatrix} -I & E_{12} \\ 0 & VrD_2 \end{pmatrix}$ is bijective and bounded, see Proposition 3.9. Since for all $\lambda \in \mathbb{C}$, the operator

$$\begin{pmatrix} sp_U \\ D_2^{-1}(I - p_U) \end{pmatrix} : W_4^2(0,a) \longrightarrow (L_2(0,a) \oplus \mathbb{C}^k) \oplus \mathbb{C}^l$$

is injective and bounded, its astriction

$$\begin{pmatrix} sp_U \\ D_2^{-1}(I - p_U) \end{pmatrix}_a : W_4^2(0,a) \longrightarrow \mathscr{D}(L(\cdot,\alpha)) \oplus \mathbb{C}^l$$

is bijective and bounded for all $\lambda \in \mathbb{C}$, see Proposition 3.10. But it is obvious that the domain of the operator $\begin{pmatrix} L(\lambda,\alpha) & 0 \\ 0 & I_{\mathbb{C}^l} \end{pmatrix}$ is $\mathscr{D}(L(\cdot,\alpha)) \oplus \mathbb{C}^l$. So it follows from Proposition 3.11 that the boundary eigenvalue operator function $T$ and the operator pencil $L(\cdot,\alpha)$ have the same resolvent set. □

## 3.3 Spectrum of the differential operators

The following proposition has been proved in [34] for the particular case where only one of the boundary conditions is $\lambda$-dependent.

**Proposition 3.13.** *The boundary eigenvalue operator function $T$ and therefore the operator pencil $L(\cdot,\alpha)$ is a Fredholm valued function with index 0. The spectrum of the Fredholm operator $T$ and therefore the Fredholm operator $L(\cdot,\alpha)$ consists of discrete eigenvalues of finite multiplicities and all eigenvalues of $L(\cdot,\alpha)$, $\alpha \geq 0$, lie in the closed upper half-plane and on the imaginary axis and are symmetric with respect to the imaginary axis.*

*Proof.* We, first, prove the statement about the location and the symmetry of the eigenvalues of the operator pencil $L(\cdot,\alpha)$, after that we prove the first part of the proposition.

Let $Y_k$ be an eigenvector corresponding to the eigenvalue $\lambda_k$ of the operator pencil $L(\lambda,\alpha)$. Then

$$\lambda_k^2(MY_k,Y_k) - i\lambda_k\alpha(KY_k,Y_k) - (A(U)Y_k,Y_k) = 0. \tag{3.22}$$

Thus

$$(\Re\lambda_k + i\Im\lambda_k)^2(MY_k,Y_k) - i(\Re\lambda_k + i\Im\lambda_k)\alpha(KY_k,Y_k) - (A(U)Y_k,Y_k) = 0. \tag{3.23}$$

It follows from (3.23) that

$$\begin{cases} ((\Re\lambda_k)^2 - (\Im\lambda_k)^2)(MY_k,Y_k) + \Im\lambda_k\alpha(KY_k,Y_k) - (A(U)Y_k,Y_k) = 0 \\ 2\Re\lambda_k\Im\lambda_k(MY_k,Y_k) - \alpha\Re\lambda_k(KY_k,Y_k) = 0. \end{cases} \tag{3.24}$$

Taking the imaginary part of equation (3.23), which is the second equation in (3.24) we obtain

$$2\Re\lambda_k\Im\lambda_k(MY_k,Y_k) - \alpha\Re\lambda_k(KY_k,Y_k) = 0.$$

This yields

$$\text{either } \Re\lambda_k = 0 \ \text{ or } 2\Im\lambda_k(MY_k,Y_k) - \alpha(KY_k,Y_k) = 0.$$

Since $M|_{\mathscr{D}(A(U))} > 0$ see Proposition 3.3, $K \geq 0$ and $\alpha \geq 0$, we obtain $\Re\lambda_k = 0$ or $\Im\lambda_k = \frac{\alpha(KY_k,Y_k)}{2(MY_k,Y_k)} \geq 0$. The symmetry follows from

$$(-\overline{\lambda})^2 M\overline{Y} - i\alpha(-\overline{\lambda})K\overline{Y} - A\overline{Y} = \overline{\lambda^2 MY - i\alpha\lambda KY - AY}$$

for all $\lambda \in \mathbb{C}$ and $Y \in \mathscr{D}(L(\cdot,\alpha))$, where $\overline{Y}$ denotes the conjugate complex of $Y$.

As the eigenvalues of the operator pencil $L(\cdot,\alpha)$ lie only in the closed upper half-plane and on the imaginary axis, its spectrum $\sigma(L(\cdot,\alpha))$ is a proper subset of $\mathbb{C}$. Thus $\rho(L(\cdot,\alpha)) \neq \emptyset$. Let

$$C(\lambda) := \begin{pmatrix} -I & E_{12} \\ 0 & VrD_2 \end{pmatrix} \quad \text{and} \quad D(\lambda) := \begin{pmatrix} sp_U \\ D_2^{-1}(I - p_U) \end{pmatrix}_a .$$

Then

$$T(\lambda) = C(\lambda) \begin{pmatrix} L(\lambda,\alpha) & 0 \\ 0 & I_{\mathbb{C}^l} \end{pmatrix} D(\lambda),$$

see Theorem 2.79. Since $C(\lambda)$ and $D(\lambda)$ are invertible, $T$ is an abstract boundary eigenvalue operator function in the sense of Section 2.5. Thus it follows from Corollary 2.105 that the boundary eigenvalue operator function $T$ and therefore the pencil operator $L(\cdot,\alpha)$ is a Fredholm operator valued function and $\rho(T) = \rho(L(\cdot,\alpha))$. As $T$ is a Fredholm operator valued function and $\rho(T) \neq \emptyset$, then it follows from Theorem 2.55 that the spectrum $\sigma(T)$ and therefore the spectrum $\sigma(L(\cdot,\alpha))$ of the Fredholm operator $L(\cdot,\alpha)$ consists of discrete eigenvalues $\lambda$. Since $T(\lambda)$ is a Fredholm operator, for each $\lambda \in \sigma(T)$, then nul $T(\lambda) =$ dim $N(T(\lambda)) < \infty$, see Definition 2.23. Because $\dim N(T(\lambda)) < \infty$, there exist a positive number $r$ such that $r = \text{nul } T(\lambda)$ and $m = \sum\limits_{j=1}^{r} m_j$ where $m_j$ is as defined in (2.14). We recall the numbers $r$ and $m$ are respectively the geometric and algebraic multiplicities of $T$ at $\lambda$, see Definition 2.58. It follows that the spectrum of the boundary eigenvalue operator function $T$ and therefore the spectrum of the operator pencil $L(\cdot,\alpha)$ consists of discrete eigenvalues of finite multiplicities.

Let $\lambda \in \rho(T)$. Then $T(\lambda)$ is invertible, so nul $T(\lambda) = 0$ and $\text{def}(T(\lambda)) = \dim R(T(\lambda))^{\perp} = 0$. Therefore ind $(T(\lambda)) =$ nul $(T(\lambda)) -$ def $(T(\lambda)) = 0$. As ind $T$ is constant, see Theorem 2.29, it follows that ind $T = 0$. □

**Lemma 3.14.** *All nonzero real eigenvalues of $L(\cdot,\alpha)$, $\alpha > 0$, (if any) are semi-simple, i.e., the corresponding eigenvectors do not possess associated vectors. All real eigenvalues of $L(\cdot,\alpha)$, $\alpha > 0$, are independent of $\alpha$.*

*Proof.* Let $\lambda_0$ be a real nonzero eigenvalue, $Y_0$ a corresponding eigenvector and assume that there is a corresponding associated vector $Y_1$. Then

$$\lambda_0^2 MY_0 - i\lambda_0 \alpha KY_0 - AY_0 = 0 \tag{3.25}$$

$$2\lambda_0 MY_0 - i\alpha KY_0 + \lambda_0^2 MY_1 - i\lambda_0 \alpha KY_1 - AY_1 = 0. \tag{3.26}$$

From (3.25) we obtain

$$\lambda_0^2 (MY_0, Y_0) - i\lambda_0 \alpha (KY_0, Y_0) - (AY_0, Y_0) = 0,$$

which shows

$$(KY_0, Y_0) = 0. \tag{3.27}$$

Due to $K \geq 0$ equation (3.27) implies

$$KY_0 = 0. \tag{3.28}$$

Then from (3.25) we deduce

$$\lambda_0^2 MY_0 - AY_0 = 0, \tag{3.29}$$

and (3.26) leads to

$$2\lambda_0 (MY_0, Y_0) - i\alpha (KY_0, Y_0) + \lambda_0^2 (MY_1, Y_0) - i\lambda_0 \alpha (KY_1, Y_0) - (AY_1, Y_0) = 0.$$

Using (3.28) this gives

$$2\lambda_0 (MY_0, Y_0) + (Y_1, \lambda_0^2 MY_0) - (Y_1, AY_0) = 0, \tag{3.30}$$

and then (3.29) implies $(MY_0, Y_0) = 0$, which contradicts $M|_{\mathscr{D}(A(U))} > 0$.

Note that (3.29) is also true for $\lambda_0 = 0$ if (3.25) is satisfied, whence it follows that any real eigenvalue $\lambda_0$ is independent of $\alpha$. $\square$

**Lemma 3.15.** *Let $\lambda = -i\tau$, $\tau > 0$, be an eigenvalue of $L(\cdot, \alpha)$, $\alpha \geq 0$ . Then $\lambda$ is semi-simple.*

*Proof.* Let $Y_0$ be an eigenvector for the eigenvalue $-i\tau$ and suppose there exists a corresponding associated vector $Y_1$. Then

$$-\tau^2 MY_0 - \tau\alpha KY_0 - AY_0 = 0 \tag{3.31}$$

$$-2i\tau MY_0 - i\alpha KY_0 - \tau^2 MY_1 - \tau\alpha KY_1 - AY_1 = 0. \tag{3.32}$$

Applying (3.32) to $Y_0$ we obtain

$$-2i\tau(MY_0, Y_0) - i\alpha(KY_0, Y_0) - \tau^2(MY_1, Y_0) - \tau\alpha(KY_1, Y_0) - (AY_1, Y_0) = 0$$

or, what is the same,

$$-2i\tau(MY_0, Y_0) - i\alpha(KY_0, Y_0) + (Y_1, -\tau^2 MY_0 - \tau\alpha KY_0 - AY_0) = 0.$$

Due to (3.31) this implies $2\tau(MY_0, Y_0) + \alpha(KY_0, Y_0) = 0$ which contradicts the inequalities $\tau > 0$, $M|_{\mathscr{D}(A(U))} > 0$, $\alpha \geq 0$, and $K \geq 0$. $\square$

**Proposition 3.16.** *Let $\lambda$ be an eigenvalue of $L(\cdot, \alpha)$, $\alpha \geq 0$. Then the geometric multiplicity of $\lambda$ is at most 2.*

*Proof.* We consider the boundary conditions: $B_j(\lambda)y = 0$, $j = 1, 2, 3, 4$, see (3.2). Define

$$\begin{aligned} f_\lambda : N(L(\lambda, \alpha)) &\longrightarrow \mathbb{C}^2 \\ y &\mapsto \begin{pmatrix} y^{[3-p_1]}(0) \\ y^{[3-p_2]}(0) \end{pmatrix}. \end{aligned}$$

Let $y \in N(L(\lambda, \alpha))$ such that $f_\lambda(y) = 0$. Then

$$\left.\begin{aligned} y^{[3-p_1]}(0) = 0 \\ y^{[3-p_2]}(0) = 0 \end{aligned}\right\}. \tag{3.33}$$

If the boundary condition $B_1(\lambda)y = 0$ is independent of $\lambda$, then $y^{[p_1]}(0) = 0$, see Theorem 3.2; however if $B_1(\lambda)y = 0$ depends on $\lambda$, since $y^{[3-p_1]}(0) = 0$, see (3.33), then $y^{[p_1]}(0) = 0$. In the same manner we have $y^{[p_2]}(0) = 0$ if the boundary condition $B_2(\lambda)y = 0$ depends or not on $\lambda$. It follows from (3.33) that $y = 0$, thus $f_\lambda$ is injective. Hence the dimension of $N(L(\lambda, \alpha))$ is at most 2 and Proposition 3.16 follows. $\square$

**Lemma 3.17.** *Let $\lambda_k(\alpha) = i\tau$, $\tau \in \mathbb{R} \setminus \{0\}$, be an eigenvalue of $L(\cdot, \alpha)$, $\alpha \geq 0$. Then:*

1. $\operatorname{Re} \dot{\lambda}_k(0) = 0$ *and* $\operatorname{Im} \dot{\lambda}_k(0) \geq 0$*; here* $\dot{}$ *means derivative with respect to* $\alpha$.

2. *If* $\tau < 0$*, then* $\operatorname{Re} \dot{\lambda}_k(\alpha) = 0$ *and* $\operatorname{Im} \dot{\lambda}_k(\alpha) \geq 0$ *for all* $\alpha \geq 0$.

3. *If* $0$ *is an eigenvalue of* $L(\cdot,\alpha)$ *for some* $\alpha \geq 0$*, then it is an eigenvalue for all* $\alpha \geq 0$*, its geometric multiplicity is the same for all* $\alpha \geq 0$*, whereas its algebraic multiplicity is the same for all* $\alpha > 0$*. If the geometric multiplicity of the eigenvalue* $0$ *is* $1$*, then the algebraic multiplicity for* $\alpha = 0$ *is* $2$*, whereas its algebraic multiplicity for* $\alpha > 0$ *is* $1$ *or* $2$*. If the geometric multiplicity of the eigenvalue* $0$ *is* $2$*, then the algebraic multiplicity for* $\alpha = 0$ *is* $4$*, whereas its algebraic multiplicity for* $\alpha > 0$ *is* $2$ *or* $3$*.*

*Proof.* First we have to justify the differentiability at $\alpha_0 = 0$ for all $\lambda = i\tau$, $\tau \neq 0$, and for $\alpha > 0$ and $\lambda = i\tau$, $\tau < 0$. If the eigenvalue is simple, then it follows from Theorem 2.72 that it is differentiable on the other hand it follows from general results that the eigenvalues are differentiable, see, e. g., Theorem 2.72, Theorem 2.73, [13], [25] or [27]. The geometric and algebraic multiplicity of a semi-simple eigenvalue are equal, see Corollary 2.69. Since the eigenvalues under consideration are semi-simple by Lemma 3.15, for $\alpha_0$ and $\tau < 0$, then they have the same geometric and algebraic multiplicity. As they are nonsimple their multiplicity is at least 2 and it follows from Proposition 3.16 that nonsimple eigenvalues have geometric and algebraic multiplicity 2. The geometric multiplicity can be at most 2. Indeed, substituting the left endpoint 0 by the right endpoint $a$ and $j \in \Theta_1^0$ by $j \in \Theta_1^a$ in the proof of Proposition 3.16, it follows that there is an eigenvector $y$ of (3.1)–(3.2) with $y^{[3-p_j]}(a) = 0$ where $j \in \Theta_1^a$ at $\lambda_k(\alpha_0)$. Since for $\alpha_0$, the boundary conditions $B_j(\lambda_k)y = 0$, $j = 1,2,3,4$, see (3.2), do not depend on $\lambda_0 = \lambda_k(\alpha_0)$, then the eigenvalue $\lambda_0 = \lambda_k(\alpha_0)$ is an eigenvalue for all $\alpha$. Let $m(\lambda,\alpha)$ denotes the characteristic determinant of (3.1)–(3.2). Then $m(\lambda,\alpha)$ depends analytically on $\lambda$ and $\alpha$, also

$$\widetilde{m}(\lambda,\alpha) = \frac{m(\lambda,\alpha)}{\lambda - \lambda_0}$$

depends analytically on $\lambda$ and $\alpha$. But now $\lambda_0$ is a simple zero of $\widetilde{m}(\lambda,\alpha)$, and thus the other eigenvalue $\lambda_{k'}(\alpha)$ depends analytically on $\alpha$ as well.

If $\lambda_k = i\tau$, $\tau \in \mathbb{R}$, is an eigenvalue of $L(\cdot,\alpha)$, $\alpha \geq 0$, then

$$-\tau^2(MY,Y) + \tau\alpha(KY,Y) - (A(U)Y,Y) = 0. \tag{3.34}$$

Here $Y$ is an eigenvector corresponding to $\lambda_k$ which depends analytically on $\alpha$. Differentiating

(3.34) with respect to $\alpha$ we obtain

$$\begin{aligned} &-\tau^2(MY,\dot{Y})+\tau\alpha(KY,\dot{Y})-(A(U)Y,\dot{Y})-\tau^2(M\dot{Y},Y)+\tau\alpha(K\dot{Y},Y) \\ &-(A(U)\dot{Y},Y)+2i\tau\dot{\lambda_k}(MY,Y)-i\alpha\dot{\lambda_k}(KY,Y)+\tau(KY,Y)=0. \end{aligned} \tag{3.35}$$

Obviously,

$$\begin{aligned} -\tau^2(MY,\dot{Y})+\tau\alpha(KY,\dot{Y})-(A(U)Y,\dot{Y}) &= \big((-\tau^2M+\tau\alpha K-A(U))Y,\dot{Y}\big) = 0 \\ -\tau^2(M\dot{Y},Y)+\tau\alpha(K\dot{Y},Y)-(A(U)\dot{Y},Y) &= \big(\dot{Y},(-\tau^2M+\tau\alpha K-A(U))Y\big) = 0. \end{aligned} \tag{3.36}$$

Substituting these equations into (3.35) we obtain

$$\big(2\tau(MY,Y)-\alpha(KY,Y)\big)\dot{\lambda_k} = i\tau(KY,Y). \tag{3.37}$$

For $\alpha_0$, as $K \geq 0$ and $M|_{\mathscr{D}(A(U))} > 0$, then $\dot{\lambda_k}(0) = \frac{i(KY,Y)}{2(MY,Y)}$ and statement 1 follows.

Since $\tau < 0$, $\alpha \geq 0$, $M|_{\mathscr{D}(A(U))} > 0$, and $K \geq 0$, it follows that

$$\dot{\lambda_k} = \frac{i\tau(KY,Y)}{2\tau(MY,Y)-\alpha(KY,Y)}, \tag{3.38}$$

so that the number $\dot{\lambda_k}$ is well defined, and statement 2 follows.

We turn our attention to statement 3. Since $L(0,\alpha) = L(0,0) = -A(U)$ is independent of $\lambda$, it follows that if 0 is an eigenvalue for some $\alpha \geq 0$, then it is eigenvalue for all $\alpha \geq 0$; also the statement about the geometric multiplicity is obvious. On the other hand, since for $\alpha_0 = 0$, $L(\cdot,0)$ is a function of $\lambda^2$, each eigenvector of $L(\lambda,0) = \lambda^2 MY_0 - AY_0$ corresponding to the eigenvalue 0 has a chain of associated vectors zero. Assume there is an eigenvector $Y_0$ corresponding to the eigenvalue 0 of $L(\cdot,0)$ which has a chain of associated vectors $Y_1$, $Y_2$, i. e.,

$$-AY_0 = 0, \quad -AY_1 = 0, \quad 2MY_0 - AY_2 = 0. \tag{3.39}$$

Taking the scalar product with $Y_0$ in the last equation and observing the first equation and the self-adjointness of $A$, we infer

$$0 = 2(MY_0,Y_0) - (AY_2,Y_0) = 2(MY_0,Y_0), \tag{3.40}$$

which gives $Y_0 = 0$ since $M|_{\mathscr{D}(A(U))} > 0$; a contradiction as $Y_0$ is an eigenvector. Thus for $\alpha_0 = 0$, the eigenvalue 0 cannot have a chain of associated vectors. Hence if the geometric multiplicity of the eigenvalue 0 is 1, then the algebraic multiplicity for $\alpha_0 = 0$ is 2, whereas if the geometric multiplicity of the eigenvalue 0 is 2, its algebraic multiplicity for $\alpha_0 = 0$ is 4. Therefore the assertions for $\alpha_0 = 0$ are proved.

Now let $\alpha > 0$. If 0 is an eigenvalue of $L(\cdot, \alpha)$ with an eigenvector $Y_0$ which has an associated vector $Y_1$, then

$$-AY_0 = 0, \quad -i\alpha KY_0 - AY_1 = 0. \tag{3.41}$$

It follows that

$$0 = -i\alpha(KY_0, Y_0) - (AY_1, Y_0) = -i\alpha(KY_0, Y_0), \tag{3.42}$$

and $K \geq 0$ implies

$$KY_0 = 0, \tag{3.43}$$

and therefore $\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y_0(x) \\ (y_0^{[3-p_j]}(a_j))_{j\in\Theta_1^a} \end{pmatrix} = KY_0 = (y_0^{[3-p_j]}(a_j))_{j\in\Theta_1^a} = 0$. This has two consequences: firstly, the solution $y_0$ of (3.1)–(3.2) is independent of $\alpha$, and secondly, the boundary conditions at $a_j$, $j \in \Theta_1^a$ are $y_0^{[p_j]}(a_j) = 0$. Thus at most one linearly independent eigenvector can have an associated vector. Assume this eigenvector $Y_0$ has a chain $Y_1$, $Y_2$ of associated vectors, i. e., (3.41) and

$$MY_0 - i\alpha KY_1 - AY_2 = 0 \tag{3.44}$$

hold. This leads to

$$0 = (MY_0, Y_0) - i\alpha(KY_1, Y_0) - (AY_2, Y_0) = (MY_0, Y_0) - i\alpha(Y_1, KY_0) - (Y_2, AY_0). \tag{3.45}$$

By (3.41) and (3.43) we thus arrive at the contradiction $(MY_0, Y_0) = 0$. Thus $Y_0$ cannot have a chain of associated vectors. Hence if the geometric multiplicity of the eigenvalue 0 is 1, then its algebraic multiplicity for $\alpha > 0$ is 1 or 2, whereas if the geometric multiplicity of the eigenvalue 0 is 2, then its algebraic multiplicity for $\alpha > 0$ is 3. This completes the proof of the assertions for $\alpha > 0$. $\square$

# Chapter 4

# Asymptotics of eigenvalues for $g = 0$ of the self-adjoint fourth order differential operators

## 4.1 Introduction

We present in Section 4.2 definitions and properties necessary to introduce Rouché's theorem given in Theorem 4.14, while we give in Section 4.3 definitions and properties on asymptotic fundamental matrices. The notions and properties presented in Section 4.2 and Section 4.3 are important for a better comprehension of the investigations conducted on the asymptotics of the eigenvalues for $g = 0$ of the boundary eigenvalue problem (3.1)–(3.2), where the boundary terms are $B_1 y = y^{[p_1]}(0)$ and $B_2 y = y^{[p_2]}(0)$, with $p_1 + p_2 \neq 0$, and for $j = 3, 4$ $B_j(\lambda)y = y^{[p_j]}(a) + i\varepsilon_j \alpha\lambda y^{[q_j]}(a)$, with $p_j + q_j = 3$ and $\varepsilon_j$ is as defined in Theorem 3.2. We present the asymptotics of the eigenvalues of the above mentioned problems in Section 4.4 to Section 4.7. For each of these problems, we use the canonical fundamental of the differential (3.1) to derive the characteristic matrix which is used to find the characteristic matrix and the characteristic function of the problem. We split the characteristic function of each of the problems in two parts, the perturbed function and the unperturbed function. We investigate the zeros of each of the unperturbed functions and prove by the mean of Rouché's applied to rectangles that there are zeros of the each of the characteristic functions which have the same asymptotics as the zeros of its corresponding unperturbed function. Applying Rouché's theorem to large squares, we prove that each of the characteristic functions of the problems (3.1)–(3.2), has the same number of zeros inside these squares as its corresponding unperturbed function.

## 4.2 Rouché's Theorem

**Definition 4.1.** A metric space $(X, d)$ is connected if the only subsets of $X$ which are both open and closed are $\emptyset$ and $X$. If $A \subset X$ then $A$ is a connected subset of $X$ if the metric space $(A, d)$ is connected. See Definition 2.1 [12, page 14].

**Proposition 4.2.** [12, page 14]. *A metric space $(X, d)$ is not connected if there are disjoint open sets $A$ and $B$ in $X$, neither of which is empty, such that $X = A \cup B$.*

**Proposition 4.3.** *A set $X \subset \mathbb{R}$ is connected if and only if $X$ is an interval.* See Proposition 2.2 [12, page 14].

**Theorem 4.4.** *An open set $G \subset \mathbb{C}$ is connected if and only if for any two points $a$, $b$ in $G$ there is a polygon from $a$ to $b$ lying entirely inside $G$.* See Theorem 2.3 [12, page 15].

**Corollary 4.5.** *If $G \subset \mathbb{C}$ is open and connected and $a$ and $b$ are points in $G$ there is a polygon $P$ in $G$ from $a$ to $b$ which is made up of line segments parallel to either the real or imaginary axis.* See Corollary 2.4 [12, page 15].

**Definition 4.6.** [12, page 40]. A region is an open connected subset of the plane.

**Definition 4.7.** A path in a region $G \subset \mathbb{C}$ is a continuous function $\gamma : [a, b] \to G$ for some interval $[a, b]$ in $\mathbb{R}$. If $\gamma'(t)$ exists for each $t$ in $[a, b]$ and $\gamma' : [a, b] \to \mathbb{C}$ is continuous, then $\gamma$ is a differentiable path. Also $\gamma$ is piecewise differentiable if there is a partition of $[a, b]$, $a = t_0 < t_1 < \cdots < t_n = b$, such that $\gamma$ is differentiable on each subinterval $[t_{j-1}, t_j]$, $\quad 1 \leq j \leq n$. See Definition 3.1 [12, page 45].

**Definition 4.8.** A function $\gamma : [a, b] \to \mathbb{C}$, for $[a, b] \subset \mathbb{R}$, is of bounded variation if there is a constant $M > 0$ such that for any partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$

$$v(\gamma; P) = \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \leq M.$$

The total variation of $\gamma$, $V(\gamma)$, is defined by

$$V(\gamma) = \sup\{v(\gamma; P) : P \text{ a partition of } [a, b]\}.$$

See Definition 1.1 [12, page 58].

**Proposition 4.9.** *If $\gamma : [a, b] \to \mathbb{C}$ is piecewise differentiable then $\gamma$ is of bounded variation and*

$$V(\gamma) = \int_a^b |\gamma'(t)| dt.$$

See Proposition 1.3 [12, page 58].

**Definition 4.10.** [12, page 62] $\gamma$ is a rectifiable path if $\gamma$ is a function of bounded variation.

**Definition 4.11.** Let $\gamma_0,\ \gamma_1 : [a, b] \to G$ be two closed rectifiable curves in a region $G$; then $\gamma_0$ is homotopic to $\gamma_1$ in $G$ if there is a continuous function $\Gamma : [0, 1] \times [0, 1] \to G$ such that

$$\begin{cases} \Gamma(s, 0) = \gamma_0(s) \quad \text{and} \quad \Gamma(s, 1) = \gamma_1(s) \quad (0 \leq s \leq 1), \\ \Gamma(0, t) = \Gamma(1, t) \quad (0 \leq t \leq 1). \end{cases} \tag{4.1}$$

See Definition 4.1 [12, page 81].

**Remark 4.12.** [12, page 81]. If $\gamma_0$ is homotopic to $\gamma_1$ in $G$ write $\gamma_0 \sim \gamma_1$.

**Definition 4.13.** If $\gamma$ is a closed rectifiable curve in $G$ then $\gamma$ is homotopic to zero ($\gamma \sim 0$) if $\gamma$ is homotopic to a constant curve in $G$. See Definition 4.8 [12, page 85].

**Theorem 4.14.** Rouché's Theorem [12, page 121]. *Suppose that $f$ and $g$ are meromorphic in a region $G$ and $\overline{B}(a; R) \subset G$. If $f$ and $g$ have no zeros or poles on the circle $\gamma = \{z : |z - a| = R\}$ and $|f(z) - g(z)| < |g(z)|$ for $z$ on $\gamma$ then*

$$Z_f - P_f = Z_g - P_g,$$

*where $Z_f$, $Z_g(P_f, P_g)$ are the number of zeros (poles) of $f$ inside $|z| = R$ counted according to multiplicity.*

## 4.3 Asymptotic fundamental matrices

**Definition 4.15.** [31, page 77]. Let $U$ be an unbounded subset of $\mathbb{C}$, $f$ be a function on $U$ with values in $M_{k,n}(\mathbb{C})$ and $g$ be a complex-valued function on $U$. We write

$$f(\lambda) = O(g(\lambda))$$

if there is a $C > 0$ such that $|f(\lambda)| \leq C|g(\lambda)|$ for $\lambda \in U$. The notation

$$f(\lambda) = o(g(\lambda))$$

means that $|f(\lambda)||g(\lambda)|^{-1} \to 0$ as $\lambda \to \infty$ in $U$. Let $a \in M_{k,n}(\mathbb{C})$. We write

$$f(\lambda) = [a]$$

if $f(\lambda) - a = o(1)$.

**Definition 4.16.** [31, page 77]. Let $U$ be an unbounded subset of $\mathbb{C}$, $g$ be a complex-valued function on $U$. Let $f(\cdot, \lambda) \in M_{k,n}(L_p(a, b))$ for $\lambda \in U$. Write

$$f(\cdot, \lambda) = \{O(g(\lambda))\}_p \quad \text{or} \quad f(\cdot, \lambda) = O(g(\lambda)) \quad \text{in} \quad M_{k,n}(L_p(a, b))$$

if there is $C > 0$ such that $|f(\cdot, \lambda)|_p \leq C|g(\lambda)|$ for $\lambda \in U$, and

$$f(\cdot, \lambda) = \{o(g(\lambda))\}_p \quad \text{or} \quad f(\cdot, \lambda) = o(g(\lambda)) \quad \text{in} \quad M_{k,n}(L_p(a, b))$$

if $|f(\cdot, a)|_p|g(\lambda)|^{-1} \to 0$ as $\lambda \to \infty$ in $U$. For $h \in M_{k,n}(L_p(a, b))$, we write

$$f(\lambda) = [h]_p$$

if $f(\cdot, \lambda) - h = \{o(1)\}_p$ or $f(\cdot, \lambda) - h = o(1)$ in $M_{k,n}(L_p(a, b))$.

In the remainder of the section we consider the first order systems of differential equations

$$y' - \widetilde{A}(\cdot, \lambda)y = 0, \tag{4.2}$$

where, for some $k \in \mathbb{N}$ and $\gamma > 0$,

$$\widetilde{A}(\cdot, \lambda) = \sum_{j=-1}^{k} \lambda^{-1} A_{-j} + \lambda^{-k-1} A^k(\cdot, \lambda) \quad (|\lambda| \geq \gamma). \tag{4.3}$$

**Assumption 4.17.** We assume that

i) $A_1 \in M_n(W_k^p(a,b))$,

ii) $A_{-j} \in M_n(W_{k-j}^p(a,b)) \quad (j = 0, \ldots, k)$,

iii) $A^k(\cdot, \lambda)$ belongs to $M_n(L_p(a,b))$ for $|\lambda| \geq \gamma$ and is bounded in $M_n(L_p(a,b))$ as $\lambda \to \infty$,

iv) $A_1$ has the diagonal form

$$A_1 = \begin{pmatrix} A_0^1 & & & & \\ & A_1^1 & 0 & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \cdot & \\ & & & & & A_l^1 \end{pmatrix},$$

where $A_\nu^1 = r_\nu I_{n_\nu},\ (\nu = 0, \ldots, l), \quad \sum_{\nu=0}^{l} n_\nu = n$, $I_{n_\nu}$ is the $n_\nu \times n_\nu$ unit matrix.

v) For $\nu, \mu \in \{0, \ldots, l\}$ there are $\phi_{\nu\mu} \in \mathbb{R}$ such that

$$r_\nu(x) - r_\mu(x) = |r_\nu(x) - r_\mu(x)| e^{i\phi_{\nu\mu}} \tag{4.4}$$

holds for all $x \in [a,b]$. Finally we assume

$$(r_\nu - r_\mu)^{-1} \in L_\infty(a,b) \quad (\nu, \mu = 0, \ldots, l;\ \nu \neq \mu). \tag{4.5}$$

For $x \in [a,b]$ and $\lambda \in \mathbb{C}$ we set

$$R_\nu(x) := \int_a^x r_\nu(\xi) d\xi \quad (\nu = 0, \ldots, l;\ x \in [a,b]),$$
$$E_\nu(x,\lambda) := exp(\lambda R_\nu(x)) I_{n_\nu} \quad (\nu = 0, \ldots, l;\ x \in [a,b]);\ \lambda \in \mathbb{C},$$

and

$$E(x,\lambda) := \begin{pmatrix} E_0(x,\lambda) & & & & \\ & E_1(x,\lambda) & 0 & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \cdot & \\ & & & & & E_l(x,\lambda) \end{pmatrix}. \tag{4.6}$$

See Assumption 2.8.1. [31, page 82].

**Remark 4.18.** [31, page 83]. For the matrices $A_j$ and $P^{[r]}$, defined below, we form the block matrices

$$A_j =: (A_{j,\nu\mu})_{\nu,\mu=0}^{l} \quad \text{and} \quad P^{[r]} =: (P_{\nu\mu}^{[r]})_{\nu,\mu=0}^{l}$$

according to the block structure of $A_1$.

**Theorem 4.19.** *Let Assumption 4.17 be satisfied.*
***A.*** *There are $P^{[r]} \in M_n(W_{k+1-r}^p(a,b))$ $(r = 0,\dots,k)$ such that*

$$P^{[0]}A_1 - A_1P^{[0]} = 0, \quad P^{[0]}(a) = I_n, \tag{4.7}$$

$$P^{[r]\prime} - \sum_{j=0}^{r} A_{-j}P^{[r-j]} + P^{[r+1]}A_1 - A_1P^{[r+1]} = 0 \quad (r = 0,\dots,k-1), \tag{4.8}$$

$$P_{\nu\nu}^{[r]\prime} - A_{0,\nu\nu}P_{\nu\nu}^{[r]} = \sum_{q=0,q\neq\nu}^{l} A_{0,\nu q}P_{q\nu}^{[r]} + \sum_{j=1}^{k}\sum_{q=0}^{l} A_{-j,\nu q}P_{q\nu}^{[r-j]} \quad (\nu = 0,\dots,l), \tag{4.9}$$

*hold.*

***B.*** *For $1 \le q \le \infty$ we set*

$$\tau_q := \begin{cases} \max\limits_{\nu,\mu=0,\nu\neq\mu}^{l} (1 + |\Re\lambda e^{i\phi_{\nu\mu}}|)^{-1+\frac{1}{q}} & if\ l > 0, \\ |\lambda|^{-1} & if\ l = 0. \end{cases} \tag{4.10}$$

*For $r \in \{0,\dots,k\}$ let the matrices $P^{[r]}$ belong to $M_n(W_{k+1-r}^p(a,b))$ and fulfill* (4.7), (4.8), (4.9). *We assert that for $|\lambda| \ge \gamma$ there is a matrix function $B_k(\cdot,\lambda) \in M_n(W_1^p(a,b))$ with the following properties:*

i) *For $|\lambda| \ge \gamma$,*

$$\widetilde{Y}(\cdot,\lambda) := \left(\sum_{r=0}^{k} \lambda^{-r}P^{[r]} + \lambda^{-k}B_k(\cdot,\lambda)\right)E(\cdot,\lambda) \tag{4.11}$$

*is a fundamental matrix of the system* (4.2)

ii) *For large $\lambda$ we have the asymptotic estimates*

$$B_k(\cdot,\lambda) = \{o(1)\}_\infty, \tag{4.12}$$

$$B_k(\cdot,\lambda) = \{O(\tau_p(\lambda))\}_\infty, \tag{4.13}$$

$$B_k(\cdot,\lambda) = \{O(\tau_\infty(\lambda))\}_\infty, \quad if \quad k > 0, \tag{4.14}$$

$$\frac{1}{\lambda}B_k'(\cdot,\lambda) = \{o(1)\}_p, \tag{4.15}$$

$$\frac{1}{\lambda}B_k'(\cdot,\lambda) = \{O(\tau_p(\lambda))\}_p. \tag{4.16}$$

iii) *Let $l > 0, k = 0$ and $p < \infty$. If $p \le \frac{3}{2}$ then we additionally assume for $\nu, \mu = 0, \dots, l$ with $\nu \neq \mu$ that $A_{0,\nu\mu} \in M_{n_\nu,n_\mu}(L_{p_{\nu\mu}}(a,b))$, where $1 \le p_{\nu\mu} \le \infty$ are such that $\frac{1}{p} + \frac{1}{p_{\nu q}} + \frac{1}{p_{q\mu}} < 2$ for all $\nu, \mu, q = 0, \dots, l$ with $\nu \neq q$ and $\mu \neq q$. Then there is a number $\varepsilon \in (0, 1 - \frac{1}{p})$ if $p > 1$ or $\varepsilon = 0$ if $p = 1$ such that*

$$B_0(\cdot,\lambda) = \left\{O\left(\max_{\nu,\mu=0,\nu\neq\mu}^{l}(1 + |\mathcal{R}(\lambda e^{i\phi_{\nu\mu}}|)^{-\frac{1}{p-\varepsilon}}\right)\right\}_p. \tag{4.17}$$

See Theorem 2.8.2 [31, pages 83-84].

**Corollary 4.20.** *We assume that the conditions* i)-iii) *in Assumption 4.17 are sharpened such that the following properties hold for some $\kappa \in \mathbb{N}$:*

i') $A_1 \in M_n(W^p_{k+\kappa}(a,b))$,

ii') $A_{-j} \in M_n(W^p_{k+\kappa-j}(a,b)) \quad (j = 0, \dots, k)$,

iii') $A^k(\cdot,\lambda) \in M_n(W^p_\kappa(a,b))$ *if* $|\lambda| \ge \gamma$ *and* $A^k(\cdot,\lambda)$ *is bounded in* $M_n(W^p_\kappa(a,b))$ *as* $\lambda \to \infty$.

*We assume that the matrix functions $P^{[r]}$ belong to $M_n(W^p_{k+1-r}(a,b))$ for all $r \in \{0, \dots, k\}$ and fulfill* (4.7), (4.8) *and* (4.9). *For $|\lambda| \ge \gamma$ let the matrix function $B_k(\cdot,\lambda)$ be defined as in part **B** of Theorem 4.19.*

*Then $P^{[r]} \in M_n(W^p_{k+1-r}(a,b))$ for $r \in \{0, \dots, k\}$ and $B_k(\cdot,\lambda) \in M_n(W^p_{k+1}(a,b))$ for $|\lambda| \ge \gamma$. We have*

$$\frac{1}{\lambda^l}B_k^{(l)}(\cdot,\lambda) = \{o(1)\}_p \tag{4.18}$$

*and*

$$\frac{1}{\lambda^l} B_k^{(l)}(\cdot, \lambda) = \{O(\tau_p(\lambda))\}_p \tag{4.19}$$

*for* $|\lambda| \geq \gamma$ *and* $l \in \{0, \dots, \kappa + 1\}$*, where* $\tau_p$ *is the function defined in* (4.10). See Corollary 2.8.3. [31, pages 93-94].

**Note 4.21.** We consider, in the following sections, the eigenvalue problem (3.1)–(3.2) with $g = 0$. Let $y_j,\ j = 1, \dots, 4$ be the canonical fundamental system of the equation (3.1), with $y_j^{(m)}(0) = \delta_{j,m+1}$ for $m = 0, \dots, 3$. We consider different cases of the boundary terms $B_1y$, $B_2y$, $B_3y$ and $B_4y$ where $B_1y$ and $B_2y$ are the following: $B_1y = y^{[p_1]}(0)$ and $B_2y = y^{[p_2]}(0)$, with $p_1 + p_2 \neq 3$ and $B_3y$ and $B_4y$ are the following: $B_3y = y^{[p_3]}(a) + i\alpha\varepsilon_3\lambda y^{[q_3]}(a)$ and $B_4y = y^{[p_4]}(a) + i\alpha\varepsilon_4\lambda y^{[q_4]}(a)$ and where the collections $U$ of the four boundary conditions of the eigenvalue problems investigated in the remainder of the chapter satisfy the conditions of Theorem 3.2. Hence the differential operator $A(U)$ associated to each of these eigenvalue problems is self-adjoint. It follows that the eigenvalues of the operator $A(U)$ and therefore the eigenvalues of the corresponding problem are real.

## 4.4 The boundary terms $B_1y$ and $B_2y$ are the following: $B_1y = y(0)$ and $B_2y = y''(0)$

In the remainder of this chapter, we choose $\alpha > 0$. For $g = 0$ the differential equation (3.1) is reduced to $y^{(4)} - \lambda^2 y = 0$. Let $y = e^{\rho x}$. Then the equation $y^{(4)} - \lambda^2 y = 0$ becomes $(\rho^4 - \lambda^2)e^{\rho x} = 0$. Hence the characteristic function, (see (2.36)), of the differential equation (3.1) for $g = 0$ is

$$\pi(\mu, \rho) = \rho^4 - \mu^4, \tag{4.20}$$

where $\mu^2 = \lambda$.

The zeros of the equation (4.20) are $-\mu,\ \mu,\ -i\mu,\ i\mu$. Thus a fundamental system and a fundamental matrix of the equation (3.1) are respectively $\{\cos(\mu x),\ \sin(\mu x),\ \cosh(\mu x),\ \sinh(\mu x)\}$

and

$$Z(x,\mu) = \begin{pmatrix} \cos(\mu x) & \sin(\mu x) & \cosh(\mu x) & \sinh(\mu x) \\ -\mu\sin(\mu x) & \mu\cos(\mu x) & \mu\sinh(\mu x) & \mu\cosh(\mu x) \\ -\mu^2\cos(\mu x) & -\mu^2\sin(\mu x) & \mu^2\cosh(\mu x) & \mu^2\sinh(\mu x) \\ \mu^3\sin(\mu x) & -\mu^3\cos(\mu x) & \mu^3\sinh(\mu x) & \mu^3\cosh(\mu x) \end{pmatrix}.$$

Let $\{y_1, y_2, y_3, y_4\}$ be the canonical fundamental system of the differential equation (3.1). Set $Y := (y_j^{(i-1)})_{i,j=1}^4$. Then there is a matrix $c \in C^4$ such that $Y(x,\mu) = Z(x,\mu)c$, see Definition 2.100, and $I_4 = Y(0,\mu) = Z(0,\mu)c$, see Theorem 2.91. Hence $c = Z(0,\mu)^{-1}$. However

$$Z(0,\mu) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \mu & 0 & \mu \\ -\mu^2 & 0 & \mu^2 & 0 \\ 0 & -\mu^3 & 0 & \mu^3 \end{pmatrix},$$

thus

$$c = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2\mu^2} & 0 \\ 0 & \frac{1}{2\mu} & 0 & -\frac{1}{2\mu^3} \\ \frac{1}{2} & 0 & \frac{1}{2\mu^2} & 0 \\ 0 & \frac{1}{2\mu} & 0 & \frac{1}{2\mu^3} \end{pmatrix}$$

and it follows that

$$\begin{cases} y_1 = \frac{1}{2}\cos(\mu x) + \frac{1}{2}\cosh(\mu x), \\ y_2 = \frac{1}{2\mu}\sin(\mu x) + \frac{1}{2\mu}\sinh(\mu x), \\ y_3 = -\frac{1}{2\mu^2}\cos(\mu x) + \frac{1}{2\mu^2}\cosh(\mu x), \\ y_4 = -\frac{1}{2\mu^3}\sin(\mu x) + \frac{1}{2\mu^3}\sinh(\mu x). \end{cases} \tag{4.21}$$

It follows from the canonical fundamental system of the equation (3.1), that

$$\begin{cases} B_1y_1 = y_1(0) = 1 \\ B_1y_2 = y_2(0) = 0 \\ B_1y_3 = y_3(0) = 0 \\ B_1y_4 = y_4(0) = 0 \end{cases} \qquad \begin{cases} B_2y_1 = y_1''(0) = 0 \\ B_2y_2 = y_2''(0) = 0 \\ B_2y_3 = y_3''(0) = 1 \\ B_2y_4 = y_4''(0) = 0 \end{cases}.$$

It follows that the characteristic matrix of this particular boundary problem is

$$M_c = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ B_3y_1 & B_3y_2 & B_3y_3 & B_3y_4 \\ B_4y_1 & B_4y_2 & B_4y_3 & B_4y_4 \end{pmatrix}. \tag{4.22}$$

The determinant of the characteristic matrix $M_c$ gives the characteristic function of the differential equation (3.2). The shape of the matrix $M_c$ leads to a reduced characteristic matrix of the boundary value problem.

The reduced characteristic matrix of the boundary value problem is

$$M = \begin{pmatrix} B_3 \\ B_4 \end{pmatrix} \begin{pmatrix} y_2 & y_4 \end{pmatrix} = \begin{pmatrix} B_3y_2 & B_3y_4 \\ B_4y_2 & B_4y_4 \end{pmatrix}. \tag{4.23}$$

It is easy to see that $\det M_c = -\det M$.

### 4.4.1 Asymptotic of the eigenvalues for $B_3y = y''(a) + i\alpha\lambda y'(a)$ and $B_4y = y^{(3)}(a) - i\alpha\lambda y(a)$

It follows from (4.23) that

$$\begin{aligned}
\det M &= B_3y_2B_4y_4 - B_4y_2B_3y_4 \\
&= (y_2''(a) + i\alpha\mu^2 y_2'(a))(y_4^{(3)}(a) - i\alpha\mu^2 y_4(a)) - (y_2^{(3)}(a) - i\alpha\mu^2 y_2(a))(y_4''(a) + i\alpha\mu^2 y_4'(a)) \\
&= y_2''(a)y_4^{(3)}(a) + \alpha^2\mu^4 y_2'(a)y_4(a) - y_2^{(3)}(a)y_4''(a) - \alpha^2\mu^4 y_2(a)y_4'(a) + i\alpha\mu^2(y_2'(a)y_4^{(3)}(a) \\
&\quad - y_2''(a)y_4(a) - y_2^{(3)}(a)y_4'(a) + y_2(a)y_4''(a)) \\
&= y_2''(a)y_4^{(3)}(a) - y_2^{(3)}(a)y_4''(a) + \alpha^2\mu^4 y_2'(a)y_4(a) - \alpha^2\mu^4 y_2(a)y_4'(a) \\
&\quad + i\alpha\mu^2(y_2'(a)y_4^{(3)}(a) - y_2''(a)y_4(a) + y_2(a)y_4''(a) - y_4'(a)y_2^{(3)}(a)).
\end{aligned} \tag{4.24}$$

But

$$\begin{cases} y_4(x) = -\frac{1}{2\mu^3}\sin(\mu x) + \frac{1}{2\mu^3}\sinh(\mu x), \\ y_4'(x) = -\frac{1}{2\mu^2}\cos(\mu x) + \frac{1}{2\mu^2}\cosh(\mu x), \\ y_4''(x) = \frac{1}{2\mu}\sin(\mu x) + \frac{1}{2\mu}\sinh(\mu x), \\ y_4^{(3)}(x) = \frac{1}{2}\cos(\mu x) + \frac{1}{2}\cosh(\mu x). \end{cases} \tag{4.25}$$

It follows from (4.21) and (4.25) that

$$\begin{cases} y_2(x) = y_4''(x), \\ y_2'(x) = y_4^{(3)}(x), \\ y_2''(x) = (y_4^{(3)})'(x) = \mu(-\frac{1}{2}\sin(\mu x) + \frac{1}{2}\sinh(\mu x)) = \mu^4 y_4(x), \\ y_2^{(3)}(x) = \mu^4 y_4'(x), \end{cases} \tag{4.26}$$

so

$$\begin{aligned} \det M &= \mu^4 y_4(a)y_4^{(3)}(a) - \mu^4 y_4'(a)y_4''(a) + \alpha^2\mu^4 y_4^{(3)}(a)y_4(a) - \alpha^2\mu^4 y_4''(a)y_4'(a) \\ &\quad + i\alpha\mu^2((y_4^{(3)}(a))^2 - \mu^4(y_4(a))^2 + (y_4''(a))^2 - \mu^4(y_4'(a))^2) \\ &= (1+\alpha^2)\mu^4(y_4(a)y_4^{(3)}(a) - y_4'(a)y_4''(a)) \\ &\quad + i\alpha\mu^2((y_4^{(3)}(a))^2 - \mu^4(y_4'(a))^2) + (y_4''(a))^2 - \mu^4((y_4(a))^2). \end{aligned} \tag{4.27}$$

But

$$\begin{aligned} y_4(a)y_4^{(3)}(a) &= \Big(-\frac{1}{2\mu^3}\sin(\mu a) + \frac{1}{2\mu^3}\sinh(\mu a)\Big)\Big(\frac{1}{2}\cos(\mu a) + \frac{1}{2}\cosh(\mu a)\Big) \\ &= -\frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a) \\ &\quad + \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a), \end{aligned} \tag{4.28}$$

$$\begin{aligned} y_4'(a)y_4''(a) &= \Big(-\frac{1}{2\mu^2}\cos(\mu a) + \frac{1}{2\mu^2}\cosh(\mu a)\Big)\Big(\frac{1}{2\mu}\sin(\mu a) + \frac{1}{2\mu}\sinh(\mu a)\Big) \\ &= -\frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) \\ &\quad + \frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a), \end{aligned} \tag{4.29}$$

$$\begin{aligned}(y_4(a))^2 &= \Big(-\frac{1}{2\mu^3}\sin(\mu a) + \frac{1}{2\mu^3}\sinh(\mu a)\Big)^2\\ &= \frac{1}{4\mu^6}\sin^2(\mu a) - \frac{1}{2\mu^6}\sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^6}\sinh^2(\mu a),\end{aligned} \tag{4.30}$$

$$\begin{aligned}(y_4'(a))^2 &= \Big(-\frac{1}{2\mu^2}\cos(\mu a) + \frac{1}{2\mu^2}\cosh(\mu a)\Big)^2\\ &= \frac{1}{4\mu^4}\cos^2(\mu a) - \frac{1}{2\mu^4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4\mu^4}\cosh^2(\mu a),\end{aligned} \tag{4.31}$$

$$\begin{aligned}(y_4''(a))^2 &= \Big(\frac{1}{2\mu}\sin(\mu a) + \frac{1}{2\mu}\sinh(\mu a)\Big)^2\\ &= \frac{1}{4\mu^2}\sin^2(\mu a) + \frac{1}{2\mu^2}\sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^2}\sinh^2(\mu a),\end{aligned} \tag{4.32}$$

$$\begin{aligned}(y_4^{(3)}(a))^2 &= \Big(\frac{1}{2}\cos(\mu a) + \frac{1}{2}\cosh(\mu a)\Big)^2\\ &= \frac{1}{4}\cos^2(\mu a) + \frac{1}{2}\cos(\mu a)\cosh(\mu a) + \frac{1}{4}\cosh^2(\mu a).\end{aligned} \tag{4.33}$$

It follows from (4.28) and (4.29) that

$$\begin{aligned}y_4(a)y_4^{(3)}(a) - y_4'(a)y_4''(a) &= -\frac{1}{2\mu^3}\sin(\mu a)\cosh(\mu a) + \frac{1}{2\mu^3}\sinh(\mu a)\cos(\mu a)\\ &= \frac{1}{2\mu^3}(\sinh(\mu a)\cos(\mu a) - \sin(\mu a)\cosh(\mu a)).\end{aligned} \tag{4.34}$$

Let

$$F(a) = (y_4^{(3)}(a))^2 - \mu^4(y_4'(a))^2) + (y_4''(a))^2 - \mu^4((y_4(a))^2. \tag{4.35}$$

Then

$$\begin{aligned}F(a) &= \frac{1}{4}\cos^2(\mu a) + \frac{1}{2}\cos(\mu a)\cosh(\mu a) + \frac{1}{4}\cosh^2(\mu a)\\ &- \mu^4\Big(\frac{1}{4\mu^4}\cos^2(\mu a) - \frac{1}{2\mu^4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4\mu^4}\cosh^2(\mu a)\Big)\\ &+ \Big(\frac{1}{4\mu^2}\sin^2(\mu a) + \frac{1}{2\mu^2}\sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^2}\sinh^2(\mu a)\Big)\\ &- \mu^4\Big(\frac{1}{4\mu^6}\sin^2(\mu a) - \frac{1}{2\mu^6}\sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^6}\sinh^2(\mu a)\Big)\\ &= \cos(\mu a)\cosh(\mu a) + \frac{1}{\mu^2}\sin(\mu a)\sinh(\mu a).\end{aligned} \tag{4.36}$$

Therefore it follows from (4.35) and (4.36) that

$$(y_4^{(3)}(a))^2 - \mu^4(y_4'(a))^2) + (y_4''(a))^2 - \mu^4((y_4(a))^2 = \cos(\mu a)\cosh(\mu a) + \frac{1}{\mu^2}\sin(\mu a)\sinh(\mu a). \tag{4.37}$$

Hence (4.27), (4.34) and (4.37) give

$$\begin{aligned}\det M &= -\frac{1}{2}\mu(1+\alpha^2)(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) + i\alpha\sin(\mu a)\sinh(\mu a)\\ &+ i\alpha\mu^2\cos(\mu a)\cosh(\mu a). \end{aligned} \tag{4.38}$$

Therefore the characteristic equation $-2i\det M = 0$ is

$$\phi(\mu) := \alpha\phi_0(\mu) + \phi_1(\mu) = 0, \tag{4.39}$$

where

$$\phi_0(\mu) = 2\mu^2\cos(\mu a)\cosh(\mu a) \tag{4.40}$$

$$\phi_1(\mu) = i(1+\alpha^2)\mu(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) + 2\alpha\sin(\mu a)\sinh(\mu a). \tag{4.41}$$

The zeros of $\phi_0$ are 0, the zeros of $\cos(\mu a)$ and the zeros of $\cosh(\mu a)$. We know that $\cos(\mu a) = 0$ if and only if $\mu = \pm(2k-1)\frac{\pi}{2a}$, $k = 1, 2, \ldots$. Since $\cosh(\mu a) = \cos(i\mu a)$, then the zeros of $\cosh(\mu a)$ are $\widetilde{\mu} = \pm i(2k-1)\frac{\pi}{2a}$, $k = 1, 2, \ldots$. Therefore the zeros of $\phi_0$ are:

$$0,\ \mu_k^{0\pm} = \pm(2k-1)\frac{\pi}{2a}, \quad \text{and } \widetilde{\mu}_k^{0\pm} = \pm i(2k-1)\frac{\pi}{2a}, \quad \text{with} \quad k = 1, 2, \ldots. \tag{4.42}$$

It can be observed that the pure imaginary nonzero zeros of $\phi_0$ are the images of the real nonzero zeros of $\phi_0$ by the rotation of angle $\frac{\pi}{2}$. Let

$$\psi_0(\mu) = \mu^2 \ \text{ and } \psi_1(\mu) = \cos(\mu a). \tag{4.43}$$

Then 0 is the only zero of $\psi_0$. It is easy to check that 0 is a zero of multiplicity 2 of $\phi_0$. On the other hand we know that for $k = 1, 2, \ldots$, $\psi_1\left(\pm(2k-1)\frac{\pi}{2a}\right) = \cos\left(\pm\left((2k-1)\frac{\pi}{2a}\right)a\right) = 0$ while $\psi_1'\left(\pm(2k-1)\frac{\pi}{2a}\right) = \left(-a\sin\left(\pm(2k-1)\frac{\pi}{2a}\right)a\right) = \pm(-1)^{k+1}a \neq 0$. Whence

$$\mu_k^{0\pm} = \pm(2k-1)\frac{\pi}{2a},\ k = 1, 2, \ldots, \quad \text{are simple zeros of } \psi_1, \text{ therefore simple zeros of } \phi_0. \tag{4.44}$$

As $\cosh(\mu a) = \cos(i\mu a)$, it follows that

$$\widetilde{\mu}_k^0 = \pm i(2k-1)\frac{\pi}{2a},\ k = 1, 2, \ldots, \quad \text{are simple zeros of } \phi_0. \tag{4.45}$$

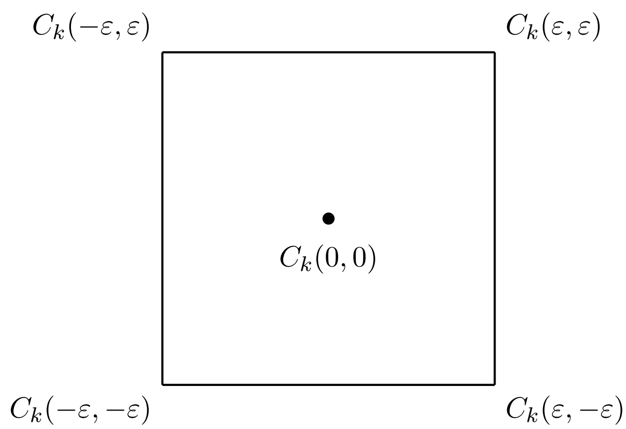
Thus the zeros of $\phi_0$ counted with multiplicity are

$$\mu_0^{0\pm} = 0,\ \mu_k^{0\pm} = \pm(2k-1)\frac{\pi}{2a}, \quad \text{and } \widetilde{\mu}_k^{0\pm} = \pm i(2k-1)\frac{\pi}{2a}, \quad \text{with} \quad k = 1, 2, \ldots. \tag{4.46}$$

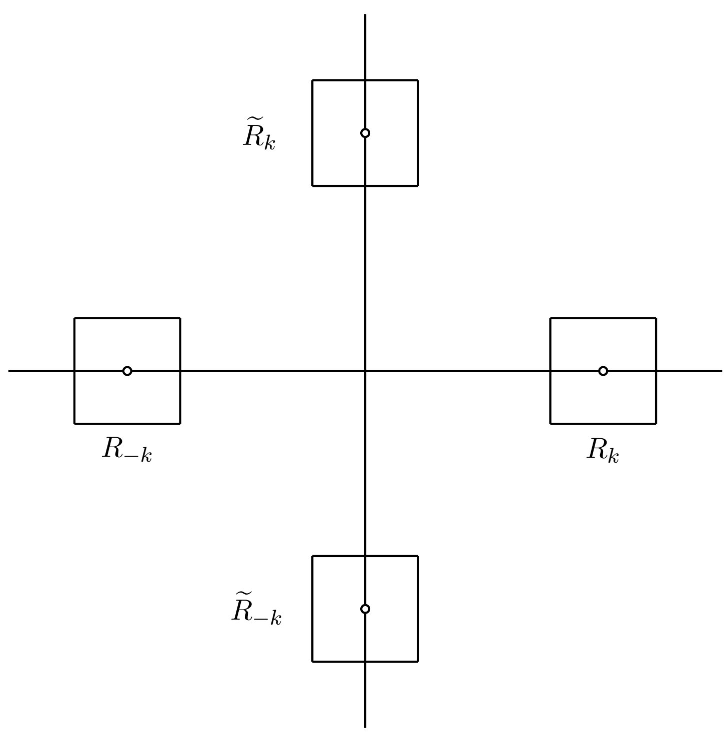
Let

$$\left.\begin{array}{l} R_k \text{ be the rectangles with vertices } (2k-1)\frac{\pi}{2a} \pm \varepsilon \pm i\varepsilon,\ k \in \mathbb{N} \text{ and } \varepsilon \in (0, \frac{\pi}{2a}) \\ \text{and } R_{-k} \text{ its symmetric image with respect to } y \text{ axis}, \widetilde{R}_k\ \widetilde{R}_{-k} \text{ the respective} \\ \text{images of the rectangles } R_k \text{ and } R_{-k} \text{ by the rotation of angle } \frac{\pi}{2}. \end{array}\right\}. \quad (4.47)$$

Let $C_k(\alpha, \beta) = (2k-1)\frac{\pi}{2a} + \alpha + i\beta$, where $\alpha,\ \beta \in \mathbb{R}$. Then the rectangle $R_k$ is the following




while the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ are as follows



Due to $\varepsilon < \frac{\pi}{2a}$, the rectangles $R_k$, $k \in \mathbb{N}$ do not intersect, as well as the rectangles $R_{-k}$, $\widetilde{R}_k$, $\widetilde{R}_{-k}$ $k \in \mathbb{N}$. We know that for $\mu \neq \pm(2k-1)\frac{\pi}{2a}$, $k = 1, 2, \ldots$, the function $\mu \mapsto \tan(\mu a)$ is periodic of period $\frac{\pi}{a}$ and it is also continuous. Since the rectangles $R_k$ are closed curves, then $\tan(\mu a)$ is bounded on the rectangles $R_k$. As the function $\mu \mapsto \tan(\mu a)$ is periodic of period $\frac{\pi}{a}$, it follows that there exists $\beta_1 > 0$ such that for all $\mu$ on the rectangles $R_k$, $k \in \mathbb{N}$

$$|\sin(\mu a)| \leq \beta_1 |\cos(\mu a)|. \tag{4.48}$$

For $\mu = x + iy$, $x,\ y \in \mathbb{R}$ and $x \neq 0$, we have

$$\begin{aligned}
|\tanh(\mu a) \pm 1| &= \left| \frac{e^{(ax+iay)} - e^{-(ax+iay)}}{e^{(ax+iay)} + e^{-(ax+iay)}} \pm 1 \right| \\
&= \left| \frac{2e^{\pm(ax+iay)}}{e^{(ax+iay)} + e^{-(ax+iay)}} \right| \\
&= \left| \frac{2}{1 + e^{\mp 2(ax+iay)}} \right|.
\end{aligned}$$

For $x < 0$, we have

$$|\tanh(\mu a) + 1| = \left| \frac{2}{1 + e^{-2(ax+iay)}} \right| \leq \frac{2}{|e^{-2ax} - 1|}$$

while for $x > 0$, we have

$$|\tanh(\mu a) - 1| = \left| \frac{2}{1 + e^{2(ax+iay)}} \right| \leq \frac{2}{|e^{2ax} - 1|}.$$

Hence for $\mu = x + iy$, $x,\ y \in \mathbb{R}$ and $x \neq 0$, we have

$$|\tanh(\mu a) \pm 1| \leq \frac{2}{e^{2a|x|} - 1}.$$

Thus for $\mu = x + iy$, where $x,\ y \in \mathbb{R}$ and $x \neq 0$, we have

$$\tanh(\mu a) \to \pm 1 \tag{4.49}$$

uniformly in $y$ as $x \to \pm\infty$.

Therefore there exist $\beta_2 \geq 1$ and $j_0 \in \mathbb{N}$ such that for all $\mu$ on the rectangles $R_k$, where $k \in \mathbb{N}$ and $k \geq j_0$,

$$|\sinh(\mu a)| \leq \beta_2 |\cosh(\mu a)|. \tag{4.50}$$

Let

$$\phi_{10}(\mu) := i\mu(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a))$$

and

$$\phi_{11}(\mu) := 2\alpha \sin(\mu a)\sinh(\mu a).$$

Then

$$\phi_1(\mu) = (1 + \alpha^2)\phi_{10}(\mu) + \phi_{11}(\mu).$$

It follows from (4.48) and (4.50) that for all $\mu$ on the rectangles $R_k$, where $k \in \mathbb{N}$ and $k \geq j_0$,

$$\begin{aligned} |1 + \alpha^2||\phi_{10}(\mu)| &= \alpha \left|\frac{1}{\alpha} + \alpha\right| |\mu||\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)| \\ &\leq \alpha(\beta_1 + \beta_2)|\mu| \left|\frac{1}{\alpha} + \alpha\right| |\cos(\mu a)\cosh(\mu a)|. \end{aligned} \tag{4.51}$$

It can be deduced from (4.48) and (4.50) that for all $\mu$ on the rectangles $R_k$, where $k \in \mathbb{N}$ and $k \geq j_0$,

$$\begin{aligned} |\phi_{11}(\mu)| &= 2\alpha|\sin(\mu a)\sinh(\mu a)| \\ &\leq 2\alpha\beta_1\beta_2|\cos(\mu a)\cosh(\mu a)|. \end{aligned} \tag{4.52}$$

For all $\mu$ on the rectangles $R_k$, we have $|\mu_k| \geq \frac{k\pi}{a}$. It can be inferred from (4.51)

$$|1 + \alpha^2||\phi_{10}(\mu)| \leq \alpha(\beta_1 + \beta_2)\frac{k\pi}{a} \left|\frac{1}{\alpha} + \alpha\right| |\cos(\mu a)\cosh(\mu a)|. \tag{4.53}$$

Thus there exists $k_0 = \frac{a}{\pi}(\beta_1 + \beta_2)(\alpha + \frac{1}{\alpha})$ such that for all $\mu$ on the rectangles $R_k$, where $k \in \mathbb{N}$ with $k \geq m_0 = \max\{j_0, k_0\}$,

$$|1 + \alpha^2||\phi_{10}(\mu)| < \alpha|\mu|^2|\cos(\mu a)\cosh(\mu a)| = \frac{1}{2}\alpha|\phi_0(\mu)|. \tag{4.54}$$

As $|\mu_k| \geq \frac{k\pi}{a}$ for all $\mu$ on the rectangles $R_k$, it can be inferred from (4.52) that there exists $\widetilde{k}_0 = \frac{a}{\pi}\sqrt{3\beta_1\beta_2}$ such that for all $\mu$ on the rectangles $R_k$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_0 = \max\{j_0, \widetilde{k}_0\}$, $|\mu|^2 \geq \frac{k^2\pi^2}{a^2} \geq \frac{a^2}{\pi^2} \times 3\beta_1\beta_2\frac{\pi^2}{a^2} = 3\beta_1\beta_2 > 2\beta_1\beta_2$. Thus for all $\mu$ on the rectangles $R_k$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_0$,

$$|\phi_{11}(\mu)| < \alpha|\mu|^2|\cos(\mu a)\cosh(\mu a)| = \frac{1}{2}\alpha|\phi_0(\mu)|. \tag{4.55}$$

It follows from (4.54) and (4.55) that, for all $\mu$ on the rectangles $R_k$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_0 = \max\{m_0, \widetilde{m}_0\}$,

$$|\phi_1(\mu)| < \alpha|\phi_0(\mu)|. \tag{4.56}$$

**Remark 4.22.** Thus it follows from Rouché's theorem that since the function $\phi_0$ has exactly one zero inside the rectangle $R_k$, then the function $\phi$ also has exactly one zero inside the rectangle $R_k$. The zero of $\phi$ inside the rectangle $R_k$ coincide with the zero of $\phi_0$ or is the asymptotic of the zero of $\phi_0$ for $k$ sufficiently large.

For all $\mu \in \mathbb{C}$, we have

$$\begin{aligned} \phi_0(-\mu) &= 2(-\mu)^2 \cos(-\mu a) \cosh(-\mu a) \\ &= 2\mu^2 \cos(\mu a) \cosh(\mu a) = \phi_0(\mu), \end{aligned} \tag{4.57}$$

while

$$\begin{aligned} \phi_1(-\mu) &= i(1+\alpha^2)(-\mu)(\sin(-\mu a) \cosh(-\mu a) - \cos(-\mu a) \sinh(-\mu a)) \\ &\quad + 2\alpha \sin(-\mu a) \sinh(-\mu a) \\ &= i(1+\alpha^2)\mu(\sin(\mu a) \cosh(\mu a) - \cos(\mu a) \sinh(\mu a)) + 2\alpha \sin(\mu a) \sinh(\mu a) \\ &= \phi_1(\mu). \end{aligned} \tag{4.58}$$

Thus it follows from (4.39), (4.57) and (4.58) that $\phi$ is an even function and therefore we have the same estimate (4.56) for all $\mu$ on the rectangle $R_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq \hat{m}_0$.

Let

$$\widetilde{\phi}_0(\mu) = 2\alpha\mu^2 \cos(\mu a) \cosh(\mu a) + 2\alpha \sin(\mu a) \sinh(\mu a) \tag{4.59}$$

and

$$\widetilde{\phi}_1(\mu) = i(1+\alpha^2)\mu(\sin(\mu a) \cosh(\mu a) - \cos(\mu a) \sinh(\mu a)). \tag{4.60}$$

Then

$$\phi(\mu) = \widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu). \tag{4.61}$$

For all $\mu \in \mathbb{C}$, we have

$$\begin{aligned} \widetilde{\phi}_0(i\mu) &= 2\alpha(i\mu)^2 \cos(i\mu a) \cosh(i\mu a) + 2\alpha \sin(i\mu a) \sinh(i\mu a) \\ &= -2\alpha\mu^2 \cos(\mu a) \cosh(\mu a) - 2\alpha \sin(\mu a) \sinh(\mu a) = -\widetilde{\phi}_0(\mu), \end{aligned} \tag{4.62}$$

while

$$\begin{aligned}\widetilde{\phi}_1(i\mu) &= i(1+\alpha^2)(i\mu)(\sin(i\mu a)\cosh(i\mu a) - \cos(i\mu a)\sinh(i\mu a)) \\ &= i(1+\alpha^2)(i\mu)(-i)(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) = \widetilde{\phi}_1(\mu). \end{aligned} \tag{4.63}$$

Hence for all $\mu \in \mathbb{C}$

$$\phi(i\mu) = -\widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu). \tag{4.64}$$

It follows from (4.61) and (4.64) that $|\phi(\mu)|$ and $|\phi(i\mu)|$ have the same upper bound $|\widetilde{\phi}_0(\mu)| + |\widetilde{\phi}_1(\mu)|$ for all $\mu$ on the rectangle $R_k$ or $R_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq \hat{m}_0$. As the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{N}$ are closed curves, then (4.39), (4.46), (4.56), Rouché's theorem and Remark 4.22 imply that there are zeros of $\phi(\mu)$ which have the same asymptotics as the zeros of $\phi_0$ where the asymptotics of the zeros of $\phi_0$ are

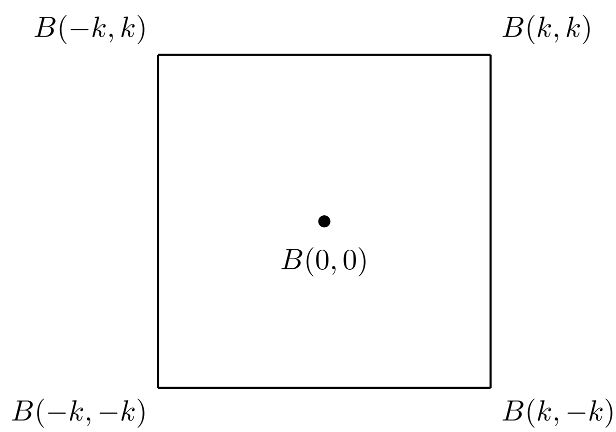
$$\left.\begin{aligned} \hat{\mu}_k^{\pm} &= \pm(2k-1)\tfrac{\pi}{2a} + o(1), \quad \text{where } k \geq \hat{m}_0 \text{ and} \\ \hat{\mu}_k^{\pm} &= \pm i(2|k|-1)\tfrac{\pi}{2a} + o(1), \ \text{where } k \leq -\hat{m}_0 \end{aligned}\right\}, \tag{4.65}$$

with $o(1) \to 0$ as $k \to \infty$.

Let

$$S_k \text{ be the square with vertices } \pm k\frac{\pi}{a} \pm ik\frac{\pi}{a}, \ k \in \mathbb{N}. \tag{4.66}$$

Let $B(p,q) = p\frac{\pi}{a} + iq\frac{\pi}{a}$, where $p, q \in \mathbb{Z}$. Then the squar $S_k$ is the following




For $\mu = k\frac{\pi}{a} + i\gamma$, $k \in \mathbb{N}$ and $\gamma \in \mathbb{R}$ we have

$$\tan(\mu a) = \tan\left(\left(k\frac{\pi}{a} + i\gamma\right)a\right) = \tan(k\pi + i\gamma a) = \tan(i\gamma a) = i\tanh(\gamma a). \tag{4.67}$$

We know that $|\tanh(\gamma a)| \leq 1$ for all $\gamma \in \mathbb{R}$. Thus it follows from (4.67) that there exists $\delta_1 > 0$ such that for $\mu = k\frac{\pi}{a} + i\gamma$, $k \in \mathbb{N}$ and $\gamma \in \mathbb{R}$

$$|\sin(\mu a)| \leq \delta_1 |\cos(\mu a)|. \tag{4.68}$$

It follows from (4.49) that there exists $\delta_2 \geq 1$ and $j_1 \in \mathbb{N}$ such that for $\mu = k\frac{\pi}{a} + i\gamma$, where $\gamma \in \mathbb{R}$ and $k \in \mathbb{N}$ with $k \geq j_1$,

$$|\sinh(\mu a)| \leq \delta_2 |\cosh(\mu a)|. \tag{4.69}$$

By interchanging tan and tanh we obtain for $\mu = \gamma + ik\frac{\pi}{a}$

$$\begin{aligned}\tanh\left(\left(\gamma + ik\frac{\pi}{a}\right)a\right) &= \tanh(\gamma a + ik\pi) = \tanh(i(-i\gamma a + k\pi)) = i\tan(-i\gamma a + k\pi)\\ &= i\tan(-i\gamma a) = -i\tan(i\gamma a) = \tanh(\gamma a),\end{aligned} \tag{4.70}$$

while

$$\tan(\mu a) = \tan\left(\left(\gamma + ik\frac{\pi}{a}\right)a\right) = \tan(\gamma a + ik\pi) = i\tanh(k\pi - i\gamma a). \tag{4.71}$$

It results from (4.49) and (4.70) that

$$\tan\left(\left(i\gamma + k\frac{\pi}{a}\right)a\right) = i\tanh\left(\left(\gamma - ik\frac{\pi}{a}\right)a\right) = i\tanh(\gamma a) \tag{4.72}$$

while it follows from (4.49) and (4.71) that

$$\tan(\mu a) = \tan\left(\left(\gamma + ik\frac{\pi}{a}\right)a\right) \to \pm i \quad \text{uniformly in} \quad \gamma \text{ as } k \to \infty. \tag{4.73}$$

Thus we obtain the same estimates (4.68) and (4.69) for $\mu = \gamma + ik\frac{\pi}{a}$, where $\gamma \in \mathbb{R}$ and $k \in \mathbb{N}$ with $k \geq j_1$. Hence there exists $\delta_1 > 0$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$

$$|\sin(\mu a)| \leq \delta_1 |\cos(\mu a)|, \tag{4.74}$$

while there exist $\delta_2 \geq 1$ and $j_1 \in \mathbb{N}$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq j_1$

$$|\sinh(\mu a)| \leq \delta_2 |\cosh(\mu a)|. \tag{4.75}$$

Therefore it can be inferred from (4.74) and (4.75) that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq j_1$, that

$$\begin{aligned}|1 + \alpha^2||\phi_{10}(\mu)| &= \alpha\left|\frac{1}{\alpha} + \alpha\right||\mu||\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)|\\ &\leq \alpha(\delta_1 + \delta_2)\left|\frac{1}{\alpha} + \alpha\right||\mu||\cos(\mu a)\cosh(\mu a)|.\end{aligned} \tag{4.76}$$

Using (4.68) and (4.69), it follows that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq j_1$,

$$\begin{aligned} |\phi_{11}(\mu)| &= 2\alpha|\sin(\mu a)\sinh(\mu a)| \\ &\leq 2\alpha\delta_1\delta_2|\cos(\mu a)\cosh(\mu a)|. \end{aligned} \tag{4.77}$$

From (4.76) it can be deduced that there exists $k_1 = \frac{a}{\pi}(\delta_1 + \delta_2)(\alpha + \frac{1}{\alpha})$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ with $k \geq m_1 = \max\{j_1, k_1\}$,

$$|1+\alpha^2||\phi_{10}(\mu)| < \alpha|\mu|^2|\cos(\mu a)\cosh(\mu a)| = \frac{1}{2}\alpha|\phi_0(\mu)| \tag{4.78}$$

and it follows from (4.77) that there exists $\widetilde{k}_1 = \frac{a}{\pi}\sqrt{3\delta_1\delta_2}$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_1 = \max\{j_1, \widetilde{k}_1\}$, $|\mu|^2 > 2\delta_1\delta_2$. Thus (4.77) and (4.78) imply that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_1$,

$$|\phi_{11}(\mu)| < \alpha|\mu|^2|\cos(\mu a)\cosh(\mu a)| = \frac{1}{2}\alpha|\phi_0(\mu)|. \tag{4.79}$$

Putting (4.78) and (4.79) together, we have for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_1 = \max\{m_1, \widetilde{m}_1\}$,

$$|\phi_1(\mu)| < \alpha|\phi_0(\mu)|. \tag{4.80}$$

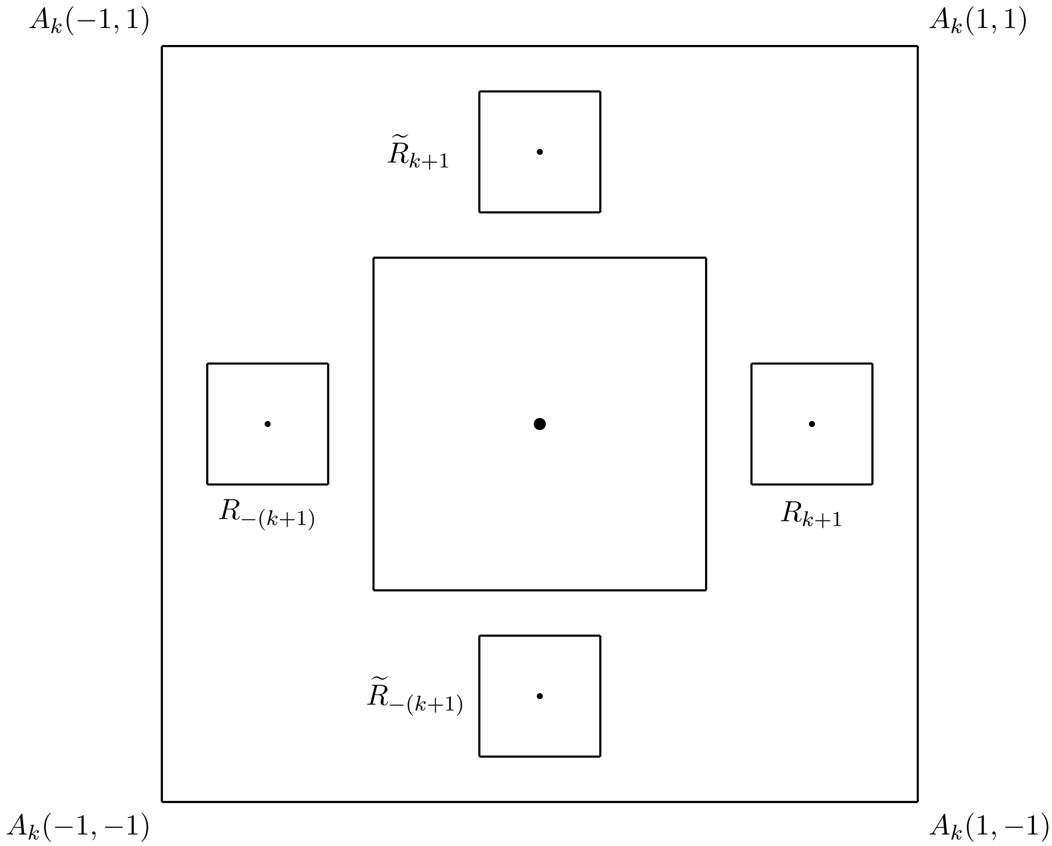
Since the square $S_k$, $k \in \mathbb{N}$ is closed, then it follows from (4.80) and from the Rouché's theorem that $\phi_0$ and $\phi$ have the same number of zeros inside the square.

**Proposition 4.23.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y(0)$, $B_2(y) = y''(0)$, $B_3 y = y''(a) + i\alpha\lambda y'(a)$ and $B_4 y = y^{(3)}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{cases} \hat{\mu}_k^{\pm} &= \pm(2k-1)\frac{\pi}{2a} + o(1), \quad \text{if } k > 0, \\ \hat{\mu}_k^{\pm} &= \pm i(2|k|-1)\frac{\pi}{2a} + o(1), \quad \text{if } k < 0. \end{cases}$$

*In particular, there is an odd number of pure imaginary eigenvalues.*

*Proof.* Let $k_1 = \max\{\hat{m}_0, \hat{m}_1\}$ and $D_{k+1}$ be the area delimited by the squares $S_k$ and $S_{k+1}$, where $k \geq k_1$. We denote by $A_k(-1,-1)$, $A_k(1,-1)$, $A_k(-1,1)$ and $A_k(1,1)$, respectivement the points $B(-k-1,-k-1)$, $B(k+1,-k-1)$, $B(-k-1,k+1)$ and $B(k+1,k+1)$. Then the domain $D_{k+1}$ is the following, where $S_k$ in the inner square and $S_{k+1}$ is the outer square,




It follows from the definitions of $R_k$, $R_{-k}$ and $S_k$, see (4.47) and (4.66) that the rectangles $R_{k+1}$ and $R_{-(k+1)}$ belong to the domain $D_{k+1}$. As the zeros $\mu_{k+1}^{0+} = \left(-\frac{\pi}{2a} + (k+1)\frac{\pi}{a}\right)$ and $\mu_{k+1}^{0-} = -\left(-\frac{\pi}{2a} + (k+1)\frac{\pi}{a}\right)$ are respectively inside the rectangles $R_{k+1}$ and $R_{-(k+1)}$, then the zeros $\mu_{k+1}^{0\pm}$ lie inside the domain $D_{k+1}$. Since the domain $D_{k+1}$ is delimited by the squares $S_k$ and $S_{k+1}$, then the respective images $\widetilde{R}_{k+1}$ and $\widetilde{R}_{-(k+1)}$ of the rectangles $R_{k+1}$ and $R_{-(k+1)}$ by the rotation of angle $\frac{\pi}{2}$ are inside the domain $D_{k+1}$. We know that the zero $\mu_{k+1}^{0+} = \left(-\frac{\pi}{2a} + (k+1)\frac{\pi}{a}\right)$ is inside the rectangle $R_{k+1}$, while the zero $\mu_{k+1}^{0-} = -\left(-\frac{\pi}{2a} + (k+1)\frac{\pi}{a}\right)$ is inside the rectangle $R_{-(k+1)}$, thus the respective images $\widetilde{\mu}_{k+1}^{0+}$ and $\widetilde{\mu}_{k+1}^{0-}$ of the zeros $\mu_{k+1}^{0+}$ and $\mu_{k+1}^{0-}$ by the rotation of angle $\frac{\pi}{2}$ are respectively inside the rectangles $\widetilde{R}_{k+1}$ and $\widetilde{R}_{-(k+1)}$. But there are only four rectangles inside the domain $D_{k+1}$, see (4.47) and the definition of the domain $D_k$, and these rectangles are $R_{k+1}$, $R_{-(k+1)}$, $\widetilde{R}_{k+1}$ and $\widetilde{R}_{-(k+1)}$. As $\mu_{k+1}^{0-} = -\mu_{k+1}^{0+}$ and $\widetilde{\mu}_{k+1}^{0-} = -\widetilde{\mu}_{k+1}^{0+}$ and the zeros $\mu_{k+1}^{0+}$, $\mu_{k+1}^{0-}$, $\widetilde{\mu}_{k+1}^{0+}$ and $\widetilde{\mu}_{k+1}^{0-}$ are respectively inside the rectangles $R_{k+1}$, $R_{-(k+1)}$, $\widetilde{R}_{k+1}$ and $\widetilde{R}_{-(k+1)}$, then it follows that there are exactly four zeros of $\phi_0$ inside the domain $D_{k+1}$. On the other hand, we know that $\phi_0$ and $\phi$ have the same number of zeros inside the square $S_k$ as well as inside the square $S_{k+1}$ for $k \geq k_1$. It follows that $\phi_0$ and $\phi$ have the same number of zeros inside the domain $D_{k+1}$. Hence there are exactly four zeros of $\phi$ inside the domain $D_{k+1}$.

The zeros of $\phi$ for $|k| \geq \hat{m}_0$ are

$$\hat{\mu}_k^{\pm}, \ k = \hat{m}_0, \hat{m}_0 + 1, \dots$$
$$- \hat{m}_0, -\hat{m}_0 - 1 \dots$$

and satisfy

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(2k-1)\frac{\pi}{2a} + o(1), & \text{where } k \geq \hat{m}_0 \text{ and} \\ \hat{\mu}_k^{\pm} = \pm i(2|k|-1)\frac{\pi}{2a} + o(1), & \text{where } k \leq -\hat{m}_0 \end{cases},$$

see (4.65).

The zeros of $\phi_0$ inside the square $S_{k_1}$, for $k_1 \geq \hat{m}_0$, are

$$\mu_0^{\pm} = 0, \ \mu_j^{0\pm} \text{ and } \widetilde{\mu}_j^{0\pm}, \ j = 1, \dots, k_1.$$

Since $\mu_j^{0\pm}$ and $\widetilde{\mu}_j^{0\pm}$ are simple zeros of $\phi_0$, while 0 is a zero of multiplicity 2, then it follows from (4.46) that the number of the zeros of $\phi_0$ inside the square $S_{k_1}$ is $2k_1 + 2k_1 + 2 = 4k_1 + 2$. As $\phi_0$ and $\phi$ have the same number of zeros inside the square $S_{k_1}$, then the number of zeros of $\phi$ inside the square $S_{k_1}$ is $4k_1 + 2$ and these zeros are

$$\hat{\mu}_0^{\pm} = \pm\left(\frac{-\pi}{2a}\right) + o(1), \ \hat{\mu}_1^{\pm} = \pm\left(\frac{\pi}{2a}\right) + o(1), \ \dots, \hat{\mu}_{k_1}^{\pm} = \pm(2k_1 - 1)\frac{\pi}{2a} + o(1),$$
$$\hat{\mu}_{-1}^{\pm} = \pm i\left(\frac{\pi}{2a}\right) + o(1), \ \hat{\mu}_{-2}^{\pm} = \pm i\left(\frac{3\pi}{2a}\right) + o(1), \dots, \hat{\mu}_{-k_1}^{\pm} = \pm i(2k_1 - 1)\frac{\pi}{2a} + o(1).$$

The functions $\phi_0$ and $\phi_1$ are even, see (4.40) and (4.41), thus $\phi$ is an even function. For the zeros $\mu$ of $\phi$, $k \in \mathbb{N}$ and $k \geq k_1$ which are inside the square $S_k$ with vertices $\pm k\frac{\pi}{a} \pm i\frac{\pi}{a}$, the numbers $\lambda = (\mu)^2$ are inside the curve $\mathscr{C}_k$, $k \in \mathbb{N}$ and $k \geq k_1$, given by the parametrization

$$\begin{cases} X = \pm\left((k\frac{\pi}{a})^2 - \gamma^2\right) \\ Y = 2\gamma k\frac{\pi}{a} \\ -k\frac{\pi}{a} \leq \gamma \leq k\frac{\pi}{a} \end{cases}. \tag{4.81}$$

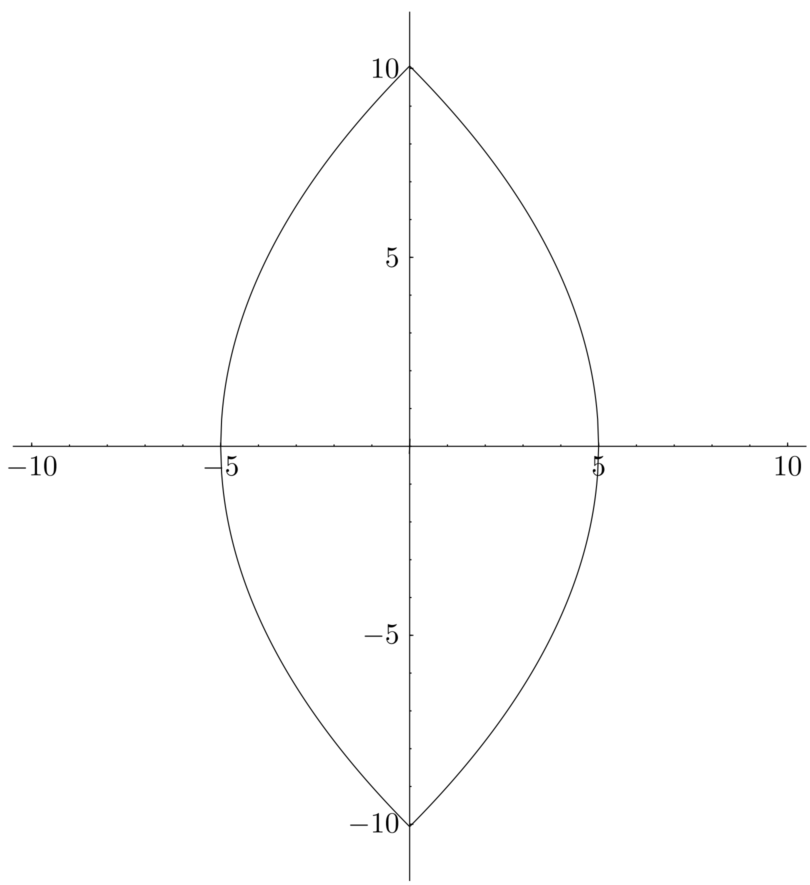
We can observe that the curve $\mathscr{C}_k$ is composed by two parabolas of respective equations

$$\begin{cases} X = \left(k\frac{\pi}{a}\right)^2 - \left(\frac{aY}{2k\pi}\right)^2, \\ -2\left(k\frac{\pi}{a}\right)^2 \leq Y \leq 2\left(k\frac{\pi}{a}\right)^2, \end{cases}$$

and

$$\begin{cases} X = -\left(k\frac{\pi}{a}\right)^2 + \left(\frac{aY}{2k\pi}\right)^2, \\ -2\left(k\frac{\pi}{a}\right)^2 \leq Y \leq 2\left(k\frac{\pi}{a}\right)^2. \end{cases}$$

The curve $\mathscr{C}_k$ is the following curve




As there are $4k_1 + 2$ zeros $\hat{\mu}_k^{\pm}$ of $\phi$ inside the square $S_{k_1}$, then it follows that the number of the eigenvalues $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ of (3.1)–(3.2) inside the curve $\mathscr{C}_{k_1}$ is $2k_1 + 1$.

Let $n$ be the number of the eigenvalues $\hat{\lambda}$ inside the curve $\mathscr{C}_{k_1}$, of the problem (3.1)–(3.2) such that $\Re(\hat{\lambda}) > 0$. It follows from Proposition 3.13 that there exist $n$ other eigenvalues $\widetilde{\lambda}$ of the problem (3.1)–(3.2) inside the curve $\mathscr{C}_{k_1}$ such that $\Re(\widetilde{\lambda}) < 0$ and for each eigenvalue $\hat{\lambda}$, there exists one and only one eigenvalue $\widetilde{\lambda}$ such that $\hat{\lambda}$ and $\widetilde{\lambda}$ are symmetric with respect to the imaginary axis. Thus there exist $2n$ eigenvalues $\lambda$ of the problem (3.1)–(3.2) inside the curve $\mathscr{C}_{k_1}$, such that $\Re(\lambda) \neq 0$. Since the eigenvalues of the problem (3.1)–(3.2) lie in the closed upper half-plane and on the imaginary axis, see Proposition 3.13, then the $2n$ eigenvalues $\lambda$ inside the curve $\mathscr{C}_{k_1}$ of the problem (3.1)–(3.2), such that $\Re(\lambda) \neq 0$, lie in the closed upper half-plane and the remaining $2(k_1 - n) + 1$ eigenvalues $\lambda$ lie on the imaginary axis. We know that the zeros $\mu$ of $\phi$ outside the square $S_{k_1}$ are inside the rectangles $R_k$ and $R_{-k}$, where $k \in \mathbb{N}$ and $k > k_1$, thus these zeros $\mu$ of $\phi$ outside the square $S_{k_1}$ are such that $\lambda = (\mu)^2$ are not pure imaginary, see (4.65). Thus the pure imaginary eigenvalues $\lambda$ of the problem (3.1)–(3.2) lie only inside the curve $\mathscr{C}_{k_1}$. Let $k_0 = k_1 - n + 1$. Then we can enumerate the eigenvalues $\hat{\lambda}_k$ such that

$$\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k} \text{ for } k = k_0, k_0 + 1, \dots$$
$$- k_0, -k_0 - 1, \dots$$

and

$$\hat{\lambda}_k \quad \text{are pure imaginary for } \; - k_0 + 1, -k_0 + 2, \dots, -1, 0, 1, \dots, k_0 - 2, k_0 - 1.$$

Thus it follows that there are $2(k_0 - 1) + 1 = 2(k_1 - n) + 1$ pure imaginary eigenvalues $\hat{\lambda}_k$. Hence there is an odd number of pure imaginary eigenvalues $\hat{\lambda}_k$. □

### 4.4.2 Asymptotic of the eigenvalues for $B_3 y = y''(a) + i\alpha\lambda y'(a)$ and $B_4 y = y(a) + i\alpha\lambda y^{(3)}(a)$

It follows from (4.23) that

$$\begin{aligned}
\det M &= B_3 y_2 B_4 y_4 - B_4 y_2 B_3 y_4 \\
&= (y_2''(a) + i\alpha\mu^2 y_2'(a))(y_4(a) + i\alpha\mu^2 y_4^{(3)}(a)) - (y_2(a) + i\alpha\mu^2 y_2^{(3)}(a))(y_4''(a) + i\alpha\mu^2 y_4'(a)) \\
&= y_2''(a) y_4(a) - y_2(a) y_4''(a) - \alpha^2\mu^4 y_2'(a) y_4^{(3)}(a) + \alpha^2\mu^4 y_2^{(3)}(a) y_4'(a) \\
&+ i\alpha\mu^2 (y_2'(a) y_4(a) + y_2''(a) y_4^{(3)}(a) - y_2^{(3)}(a) y_4''(a) - y_4'(a) y_2(a)). \qquad (4.82)
\end{aligned}$$

We can infer from (4.26) that

$$\begin{aligned}
\det M &= \mu^4(y_4(a))^2 - (y_4''(a))^2 - \alpha^2\mu^4(y_4^{(3)}(a))^2 + \alpha^2\mu^8(y_4'(a))^2 \\
&+ i\alpha\mu^2(y_4^{(3)}(a)y_4(a) + \mu^4 y_4(a)y_4^{(3)}(a) - \mu^4 y_4'(a)y_4''(a) - y_4'(a)y_4''(a)) \\
&= \alpha^2\mu^8(y_4'(a))^2 + \mu^4(y_4(a))^2 - \alpha^2\mu^4(y_4^{(3)}(a))^2 - (y_4''(a))^2 \\
&+ i\alpha\mu^2(1+\mu^4)(y_4^{(3)}(a)y_4(a) - y_4'(a)y_4''(a)). \qquad (4.83)
\end{aligned}$$

Let

$$G(a) = \alpha^2\mu^8(y_4'(a))^2 - \alpha^2\mu^4(y_4^{(3)}(a))^2 + \mu^4(y_4(a))^2 - (y_4''(a))^2. \qquad (4.84)$$

Then it follows from (4.30)–(4.33) that

$$\begin{aligned}
G(a) &= \frac{\alpha^2}{4}\mu^4\left(\cos^2(\mu a) - 2\cos(\mu a)\cosh(\mu a) + \cosh^2(\mu a)\right) \\
&- \frac{\alpha^2}{4}\mu^4\left(\cos^2(\mu a) + 2\cos(\mu a)\cosh(\mu a) + \cosh^2(\mu a)\right) \\
&+ \frac{1}{4\mu^2}\left(\sin^2(\mu a) - 2\sin(\mu a)\sinh(\mu a) + \sinh^2(\mu a)\right) \\
&- \frac{1}{4\mu^2}\left(\sin^2(\mu a) + 2\sin(\mu a)\sinh(\mu a) + \sinh^2(\mu a)\right) \\
&= -\alpha^2\mu^4\cos(\mu a)\cosh(\mu a) - \frac{1}{\mu^2}\sin(\mu a)\sinh(\mu a). \qquad (4.85)
\end{aligned}$$

We recall that

$$y_4(a)y_4^{(3)}(a) - y_4'(a)y_4''(a) = \frac{1}{2\mu^3}\left(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)\right), \qquad (4.86)$$

see (4.34). It follows from (4.83)–(4.86) that

$$\begin{aligned}
\det M &= -\alpha^2\mu^4\cos(\mu a)\cosh(\mu a) + \frac{i\alpha}{2}\mu^3(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) \\
&+ \frac{i\alpha}{2\mu}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) - \frac{1}{\mu^2}\sin(\mu a)\sinh(\mu a). \qquad (4.87)
\end{aligned}$$

The characteristic equation $-2\det M = 0$ is

$$\phi(\mu) = 2\alpha^2\phi_0(\mu) + \phi_1(\mu) = 0, \qquad (4.88)$$

where

$$\phi_0(\mu) = \mu^4\cos(\mu a)\cosh(\mu a) \qquad (4.89)$$

$$\begin{aligned}
\phi_1(\mu) &= i\alpha\mu^3(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \\
&+ \frac{i\alpha}{\mu}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) + \frac{2}{\mu^2}\sin(\mu a)\sinh(\mu a). \qquad (4.90)
\end{aligned}$$

The zeros of $\phi_0$ are

$$\left.\begin{array}{l} \mu_{-1}^{0\pm} = 0,\ \mu_1^{0\pm} = 0,\ \mu_k^{0\pm} = \pm(2k-3)\frac{\pi}{2a} \\ \text{and } \widetilde{\mu}_{-k}^{0\pm} = \pm i(2k-3)\frac{\pi}{2a},\ \text{with } k = 2, 3 \ldots, \end{array}\right\} \tag{4.91}$$

see (4.40), (4.46) and (4.89). We recall that 0 is a zero of multiplicity 4 of $\phi_0$ while $\mu_k^{0\pm} = \pm(2k-3)\frac{\pi}{2a}$ and $\widetilde{\mu}_{-k}^{0\pm} = \pm i(2k-3)\frac{\pi}{2a}$, $k = 2, 3, \ldots$, are its simple zeros.

Let $R_k$, $R_{-k}$, $\widetilde{R}_k$, $\widetilde{R}_{-k}$, $k \in \mathbb{N}$ be the rectangles defined in (4.47). Let

$$\phi_{10}(\mu) = i\alpha\mu^3(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)), \tag{4.92}$$

$$\phi_{11}(\mu) = \frac{i\alpha}{\mu}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)), \tag{4.93}$$

$$\phi_{12}(\mu) = \frac{2}{\mu^2}\sin(\mu a)\sinh(\mu a). \tag{4.94}$$

Then

$$\phi_1(\mu) = \phi_{10}(\mu) + \phi_{11}(\mu) + \phi_{12}(\mu).$$

It follows from (4.48) and (4.50) that there exist $\beta_1 > 0$, $\beta_2 \geq 1$ and $j_0 \in \mathbb{N}$, such that for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq j_0$,

$$\begin{aligned} |\phi_{10}(\mu)| &\leq \frac{\alpha^2}{\alpha}(\beta_1+\beta_2)|\mu|^3|\cos(\mu a)\cosh(\mu a)|, \\ &= \frac{3(\beta_1+\beta_2)}{\alpha|\mu|}\frac{\alpha^2}{3}|\mu|^4|\cos(\mu a)\cosh(\mu a)| \end{aligned} \tag{4.95}$$

$$\begin{aligned} |\phi_{11}(\mu)| &\leq \frac{\alpha^2}{|\mu|\alpha}(\beta_1+\beta_2)|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{3(\beta_1+\beta_2)}{\alpha|\mu|^5}\frac{\alpha^2}{3}|\mu|^4|\cos(\mu a)\cosh(\mu a)| \end{aligned} \tag{4.96}$$

and

$$\begin{aligned} |\phi_{12}(\mu)| &\leq \frac{2\alpha^2}{|\mu|^2\alpha^2}\beta_1\beta_2|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{6\beta_1\beta_2}{\alpha^2|\mu|^6}\frac{\alpha^2}{3}|\mu|^4|\cos(\mu a)\cosh(\mu a)|. \end{aligned} \tag{4.97}$$

It can be deduced from (4.95) that there exists $k_0 = \frac{4a}{\alpha\pi}(\beta_1+\beta_2)$ such that for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ with $k \geq m_0 = \max\{j_0, k_0\}$,

$$|\phi_{10}(\mu)| < \frac{\alpha^2}{3}|\phi_0(\mu)|, \tag{4.98}$$

from (4.96) that there exists $\widetilde{k}_0 = \frac{a}{\pi}\sqrt[5]{\frac{4(\beta_1+\beta_2)}{\alpha}}$ such that for all $\mu$ on the rectangles $R_k$ or $R_{-k}$, where $k \in \mathbb{N}$ with $k \geq \widetilde{m}_0 = \max\{j_0, \widetilde{k}_0\}$,

$$|\phi_{11}(\mu)| < \frac{\alpha^2}{3}|\phi_0(\mu)| \tag{4.99}$$

and from (4.97) that there exists $\hat{k}_0 = \frac{a}{\pi}\sqrt[6]{\frac{7\beta_1\beta_2}{\alpha^2}}$ such that for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ with $k \geq \hat{m}_0 = \max\{j_0, \hat{k}_0\}$,

$$|\phi_{12}(\mu)| < \frac{\alpha^2}{3}|\phi_0(\mu)|. \tag{4.100}$$

It can be inferred from (4.98), (4.99) and (4.100) that, for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$ where $k \in \mathbb{N}$ and $k \geq \breve{m}_0 = \{m_0, \widetilde{m}_0, \hat{m}_0\}$,

$$|\phi_1(\mu)| < \alpha^2|\phi_0(\mu)|. \tag{4.101}$$

As the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{N}$ are closed curves, 0 is a zero of multiplicity 4, $\mu_k^{0\pm}$ and $\widetilde{\mu}_k^{0\pm}$ are simple zeros of $\phi_0$, then it follows from (4.88), (4.101) and Rouché's theorem that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$, where the asymptotics of the zeros of $\phi_0$ are

$$\left.\begin{array}{l} \hat{\mu}_k^{\pm} = \pm(2k-3)\frac{\pi}{2a} + o(1), \text{ where } k \in \mathbb{Z}, k \geq \breve{m}_0 \text{ and} \\ \hat{\mu}_k^{\pm} = \pm i(2|k|-3)\frac{\pi}{2a} + o(1), \text{ where } k \in \mathbb{Z} \text{ and } k \leq -\breve{m}_0 \end{array}\right\}, \tag{4.102}$$

with $o(1) \to 0$ as $|k| \to \infty$, see (4.91).

Let $S_k$, $k \in \mathbb{N}$ be the square defined in (4.66). It follows from (4.74) and (4.75) that there exist $\delta_1$, $\delta_2$ and $j_1 \in \mathbb{N}$ such that for all $\mu$ on the squares $S_k$, where $k \geq j_1$,

$$\begin{aligned} |\phi_{10}(\mu)| &\leq \frac{\alpha^2}{\alpha}(\delta_1+\delta_2)|\mu|^3|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{3(\delta_1+\delta_2)}{\alpha|\mu|}\frac{\alpha^2}{3}|\mu|^4|\cos(\mu a)\cosh(\mu a)|, \end{aligned} \tag{4.103}$$

$$\begin{aligned} |\phi_{11}(\mu)| &\leq \frac{\alpha^2}{|\mu|\alpha}(\delta_1+\delta_2)|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{3(\delta_1+\delta_2)}{\alpha|\mu|^5}\frac{\alpha^2}{3}|\mu|^4|\cos(\mu a)\cosh(\mu a)| \end{aligned} \tag{4.104}$$

and

$$|\phi_{12}(\mu)| \le \frac{2\alpha^2}{|\mu|^2\alpha^2}\delta_1\delta_2|\cos(\mu a)\cosh(\mu a)|$$
$$= \frac{6\delta_1\delta_2}{|\mu|^6\alpha^2}\frac{\alpha^2}{3}|\mu|^4|\cos(\mu a)\cosh(\mu a)|. \tag{4.105}$$

It can be inferred from (4.103) that there exists $k_1 = \frac{4a}{\alpha\pi}(\delta_1 + \delta_2)$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $|k| \ge m_1 = \max\{j_1, k_1\}$,

$$|\phi_{10}(\mu)| < \frac{\alpha^2}{3}|\phi_0(\mu)|, \tag{4.106}$$

from (4.104) that there exist $\widetilde{k}_1 = \frac{a}{\pi}\sqrt[5]{\frac{4(\delta_1+\delta_2)}{\alpha}}$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \ge \widetilde{m}_1 = \max\{j_1, \widetilde{k}_1\}$,

$$|\phi_{11}(\mu)| < \frac{\alpha^2}{3}|\phi_0(\mu)| \tag{4.107}$$

and from (4.105) that there exist $\hat{k}_1 = \frac{a}{\pi}\sqrt[6]{\frac{7\delta_1\delta_2}{\alpha^2}}$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \ge \hat{m}_1 = \max\{j_1, \hat{k}_1\}$,

$$|\phi_{12}(\mu)| < \frac{\alpha^2}{3}|\phi_0(\mu)|. \tag{4.108}$$

Let $\breve{m}_1 = \max\{m_1, \widetilde{m}_1, \hat{m}_1\}$. It can be concluded from (4.106), (4.107) and (4.108) that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \ge \breve{m}_1$

$$|\phi_1(\mu)| < \alpha^2|\phi_0(\mu)|. \tag{4.109}$$

As the square $S_k$ is a closed curve, then it follows from (4.88), (4.109) and Rouché's theorem that $\phi_0$ and $\phi$ have the same number of zeros inside the square.

**Remark 4.24.** Let $k_1 = \max\{\breve{m}_0, \breve{m}_1\}$. As 0 is a zero of multiplicity 4 of $\phi_0$ and $\mu_k^{0+}$, $\mu_k^{0-}$, $\widetilde{\mu}_k^{0+}$ and $\widetilde{\mu}_k^{0-}$, $k = 2, 3, \ldots$, are simple zeros, then the number of zeros of $\phi_0$ and therefore of $\phi$ inside the square $S_k$, where $k \in \mathbb{N}$ and $k \ge k_1$ is $4k$.

**Proposition 4.25.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2)*, where $B_1(y) = y(0)$, $B_2(y) = y''(0)$, $B_3 y = y''(a) + i\alpha\lambda y'(a)$ and $B_4 y = y(a) + i\alpha\lambda y^{(3)}(a)$, counted with multiplicity, can be enumerated in such a way that* *the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(2k-3)\frac{\pi}{2a} + o(1), & if\ k > 0, \\ \hat{\mu}_k^{\pm} = \pm i(2|k|-3)\frac{\pi}{2a} + o(1), & if\ k < 0. \end{cases}$$

*In particular, there is an even number of pure imaginary eigenvalues.*

*Proof.* Since $S_k$, $k \in \mathbb{N}$ is the square defined in (4.66), we consider the curve $\mathscr{C}_k$ defined in (4.81). The $4k_1$ zeros $\hat{\mu}_k^{\pm}$ of $\phi$ inside the square $S_{k_1}$ are the following:

$$\hat{\mu}_1^{\pm} = \pm\left(\frac{-\pi}{2a}\right) + o(1),\ \hat{\mu}_2^{\pm} = \pm\left(\frac{\pi}{2a}\right) + o(1), \ldots, \hat{\mu}_{k_1}^{\pm} = \pm(2k_1-3)\frac{\pi}{2a} + o(1),$$
$$\hat{\mu}_{-1}^{\pm} = \pm i\left(\frac{-\pi}{2a}\right) + o(1),\ \hat{\mu}_{-2}^{\pm} = \pm i\left(\frac{\pi}{2a}\right) + o(1), \ldots, \hat{\mu}_{-(k_1)}^{\pm} = \pm i(2k_1-3)\frac{\pi}{2a} + o(1),$$

thus

$$\hat{\mu}_k^{\pm} = \pm(2k-3)\frac{\pi}{2a} + o(1),\ \text{where } 1 \leq k \leq k_1 + 1,$$
$$\hat{\mu}_k^{\pm} = \pm i(2|k|-3)\frac{\pi}{2a} + o(1),\ \text{where } -k_1 - 1 \leq k \leq -1.$$

As there are $4k_1 + 4$ zeros $\hat{\mu}_k^{\pm}$ of $\phi$ inside the square $S_{k_1}$, then it follows that the number of eigenvalues $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ of the problem (3.1)–(3.2) inside the the curve $\mathscr{C}_{k_1}$ is $4k_1 + 4$.

Using the same approach as in the proof of Proposition 4.23, we can prove that the zeros of $\phi$ for $|k| \geq k_1$ are

$$\hat{\mu}_k^{\pm},\ k = k_1, k_1+1, \ldots$$
$$-k_1, -k_1-1, \ldots$$

and satisfy

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(2k-3)\frac{\pi}{2a} + o(1),\ \text{where } k \geq k_1, \\ \hat{\mu}_k^{\pm} = \pm i(2|k|-3)\frac{\pi}{2a} + o(1),\ \text{where } k \leq -k_1, \end{cases}$$

see (4.102).

Let $n$ be the number of the eigenvalues $\hat{\lambda}$ inside the curve $\mathscr{C}_{k_1}$, of the problem (3.1)–(3.2) such that $\Re(\hat{\lambda}) > 0$. It follows from Proposition 3.13 that there exist $n$ other eigenvalues $\widetilde{\lambda}$

of the problem (3.1)–(3.2) inside the curve $\mathscr{C}_{k_1}$ such that $\Re(\widetilde{\lambda}) < 0$ and for each eigenvalue $\hat{\lambda}$, there exists one and only one eigenvalue $\widetilde{\lambda}$ such that $\hat{\lambda}$ and $\widetilde{\lambda}$ are symmetric with respect to the imaginary axis. Thus there exist $2n$ eigenvalues $\lambda$ of the problem (3.1)–(3.2) inside the curve $\mathscr{C}_{k_1}$, such that $\Re(\lambda) \neq 0$. Since the eigenvalues of the problem (3.1)–(3.2) lie in the closed upper half-plane and on the imaginary axis, see Proposition 3.13, then the $2n$ eigenvalues $\lambda$ inside the curve $\mathscr{C}_{k_1}$ of the problem (3.1)–(3.2), such that $\Re(\lambda) \neq 0$, lie in the closed upper half-plane and the remaining $2(k_1 - n) + 2$ eigenvalues $\lambda$ lie on the imaginary axis. Let $k_0 = k_1 - n + 2$. Then we can enumerate the eigenvalues $\hat{\lambda}_k$ such that

$$\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k} \text{ for } k = k_0, k_0 + 1, \dots$$
$$- k_0, -k_0 - 1, \dots$$

and

$$\hat{\lambda}_k \ \text{ are pure imaginary for } \ -k_0 + 1, -k_0 + 2, \dots, -1, 1, 2. \dots, k_0 - 2, k_0 - 1.$$

Thus it follows that there are $2(k_0 - 1) = 2(k_1 - n) + 2$ pure imaginary eigenvalues $\hat{\lambda}_k$. Hence there is an even number of pure imaginary eigenvalues $\hat{\lambda}_k$. $\square$

### 4.4.3 Asymptotic of the eigenvalues for $B_3 y = y'(a) - i\alpha\lambda y''(a)$ and $B_4 y = y^{(3)}(a) - i\alpha\lambda y(a)$

We have from (4.23),

$$\begin{aligned}
\det M &= (B_3 y_2 B_4 y_4 - B_4 y_2 B_3 y_4) \\
&= (y_2'(a) - i\alpha\mu^2 y_2''(a))(y_4^{(3)}(a) - i\alpha\mu^2 y_4(a)) - (y_2^{(3)}(a) - i\alpha\mu^2 y_2(a))(y_4'(a) - i\alpha\mu^2 y_4''(a)) \\
&= y_4^{(3)}(a) y_2'(a) - \alpha^2\mu^4 y_4(a) y_2''(a) - y_4'(a) y_2^{(3)}(a) + \alpha^2\mu^4 y_4''(a) y_2(a) \\
&\quad - i\alpha\mu^2 (y_4(a) y_2'(a) + y_2''(a) y_4^{(3)}(a) - y_4'(a) y_2(a) - y_4''(a) y_2^{(3)}(a)).
\end{aligned} \tag{4.110}$$

Using (4.26), we have

$$
\begin{aligned}
\det M &= B_3 y_2 B_4 y_4 - B_4 y_2 B_3 y_4 \\
&= -\mu^4 (y_4'(a))^2 + \alpha^2 \mu^4 (y_4''(a))^2 + (y_4^{(3)}(a))^2 - \alpha^2 \mu^8 (y_4(a))^2 \\
&+ i\alpha\mu^2 (y_4''(a) y_4'(a) + \mu^4 y_4'(a) y_4''(a) - y_4^{(3)}(a) y_4(a) - \mu^4 y_4(a) y_4^{(3)}(a)) \\
&= -\mu^4 (y_4'(a))^2 + \alpha^2 \mu^4 (y_4''(a))^2 + (y_4^{(3)}(a))^2 - \alpha^2 \mu^8 (y_4(a))^2 \\
&+ i\alpha\mu^2 (1+\mu^4)(y_4''(a) y_4'(a) - y_4^{(3)}(a) y_4(a)). \qquad (4.111)
\end{aligned}
$$

Let

$$H_0(a) = -\mu^4 (y_4'(a))^2 + \alpha^2 \mu^4 (y_4''(a))^2 + (y_4^{(3)}(a))^2 - \alpha^2 \mu^8 (y_4(a))^2 \qquad (4.112)$$

$$H_1(a) = i\alpha\mu^2 (1+\mu^4)(y_4''(a) y_4'(a) - y_4^{(3)}(a) y_4(a)). \qquad (4.113)$$

Then it follows from (4.30)–(4.33) that

$$
\begin{aligned}
H_0(a) = &-\mu^4 \left( \frac{1}{4\mu^4} \cos^2(\mu a) - \frac{1}{2\mu^4} \cos(\mu a)\cosh(\mu a) + \frac{1}{4\mu^4} \cosh^2(\mu a) \right) \\
&+ \alpha^2 \mu^4 \left( \frac{1}{4\mu^2} \sin^2(\mu a) + \frac{1}{2\mu^2} \sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^2} \sinh^2(\mu a) \right) \\
&+ \frac{1}{4} \cos^2(\mu a) + \frac{1}{2} \cos(\mu a)\cosh(\mu a) + \frac{1}{4} \cosh^2(\mu a) \\
&- \alpha^2 \mu^8 \left( \frac{1}{4\mu^6} \sin^2(\mu a) - \frac{1}{2\mu^6} \sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^6} \sinh^2(\mu a) \right) \\
&= \alpha^2 \mu^2 \sin(\mu a)\sinh(\mu a) + \cos(\mu a)\cosh(\mu a). \qquad (4.114)
\end{aligned}
$$

Using (4.34) it is easy to check that

$$
\begin{aligned}
H_1(a) = &\frac{i\alpha}{2} \mu^3 (\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \\
&+ \frac{i\alpha}{2\mu} (\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)). \qquad (4.115)
\end{aligned}
$$

Thus it follows from (4.111)–(4.115) that

$$
\begin{aligned}
\det M &= \alpha^2 \mu^2 \sin(\mu a)\sinh(\mu a) + \cos(\mu a)\cosh(\mu a) \\
&+ \frac{i\alpha}{2} \mu^3 (\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \\
&+ \frac{i\alpha}{2\mu} (\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)). \qquad (4.116)
\end{aligned}
$$

Hence the characteristic equation $-2i \det M = 0$ is

$$\phi(\mu) = \alpha\phi_0(\mu) + \phi_1(\mu) = 0, \qquad (4.117)$$

where

$$\phi_0(\mu) = \mu^3(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \tag{4.118}$$

$$\begin{aligned}\phi_1(\mu) &= -2i\alpha^2\mu^2\sin(\mu a)\sinh(\mu a) - 2i\cos(\mu a)\cosh(\mu a)\\ &\quad + \frac{\alpha}{\mu}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)).\end{aligned} \tag{4.119}$$

For all $\mu \in \mathbb{C}$, we have

$$\phi(-\mu) = \alpha\phi_0(-\mu) + \phi_1(-\mu). \tag{4.120}$$

But

$$\begin{aligned}\phi_0(-\mu) &= (-\mu)^3(\sin(-\mu a)\cosh(-\mu a) - \cos(-\mu a)\sinh(-\mu a))\\ &- \mu^3(-\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a))\\ &= \mu^3(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) = \phi_0(\mu),\end{aligned} \tag{4.121}$$

while

$$\begin{aligned}\phi_1(-\mu) &= -2i\alpha^2(-\mu)^2\sin(-\mu a)\sinh(-\mu a) - 2i\cos(-\mu a)\cosh(-\mu a)\\ &+ \frac{\alpha}{-\mu}(\sin(-\mu a)\cosh(-\mu a) - \cos(-\mu a)\sinh(-\mu a))\\ &= -2i\alpha^2\mu^2\sin(\mu a)\sinh(\mu a) - 2i\cos(\mu a)\cosh(\mu a)\\ &+ \frac{\alpha}{-\mu}(-\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a))\\ &= -2i\alpha^2\mu^2\sin(\mu a)\sinh(\mu a) - 2i\cos(\mu a)\cosh(\mu a)\\ &+ \frac{\alpha}{\mu}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) = \phi_1(\mu).\end{aligned} \tag{4.122}$$

Thus the functions $\phi_0$ and $\phi_1$ are even functions, and therefore the function $\phi$ is even.

The zeros of $\phi_0$ are the zeros of $\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)$. Assume that $\cos(\mu a) = 0$, then $\phi_0(\mu) = 0$ implies $\sin(\mu a)\cosh(\mu a) = 0$. Hence $\sin(\mu a) = 0$ or $\cosh(\mu a) = 0$. However for $\cos(\mu a) = 0$, $\mu = \frac{(2k+1)\pi}{2a}$, where $k \in \mathbb{Z}$. Thus $\cosh(\mu a) = \frac{e^{(2k+1)\frac{\pi}{2}} + e^{-(2k+1)\frac{\pi}{2}}}{2} \neq 0$ and $\sin(\mu a) = \sin((2k+1)\frac{\pi}{2}) = \pm 1 \neq 0$. Therefore $\phi_0(\mu) \neq 0$, which is a contradiction. On the other hand if $\cosh(\mu a) = 0$, then $\phi_0(\mu) = 0$ implies $\cos(\mu a)\sinh(\mu a) = 0$. Hence $\cos(\mu a) = 0$ or $\sinh(\mu a) = 0$. However for $\cosh(\mu a) = 0$, $\mu = \frac{-i(2k+1)\pi}{2a}$, where $k \in \mathbb{Z}$. Thus $\cos(\mu a) = \frac{e^{(2k+1)\frac{\pi}{2}} + e^{-(2k+1)\frac{\pi}{2}}}{2} \neq 0$ and $\sinh(\mu a) = -\sin((2k+1)\frac{\pi}{2}) = \pm 1 \neq 0$. Therefore $\phi_0(\mu) \neq 0$, which is a contradiction. Whence $\phi_0(\mu) = 0$ implies that $\cos(\mu a) \neq 0$ and $\cosh(\mu a) \neq 0$. Therefore for $\cos(\mu a) \neq 0$ and $\cosh(\mu a) \neq 0$,

$$\phi_0(\mu) = \mu^3 \cos(\mu a) \cosh(\mu a)(\tan(\mu a) - \tanh(\mu a)),$$

hence the nonzero zeros of $\phi_0$ are given by those $\mu \neq 0$ for which $\tan(\mu a) = \tanh(\mu a)$. As $\tan'(x) \geq 1$ and $\tanh'(x) = \frac{1}{\cosh^2(x)} < 1$ for $x \in \mathbb{R}\backslash\{0\}$ the function $\tan(x) - \tanh(x)$ is increasing with positive derivative on each interval $((k - \frac{1}{2})\pi, (k + \frac{1}{2})\pi)$, with the exception of the point 0. On each of these intervals, the function moves from $-\infty$ to $\infty$, so we have exactly one simple zero $\hat{\mu}_k^{\pm}$ of $\tan(\mu a) - \tanh(\mu a)$ in each interval $((\pm k - \frac{1}{2})\pi, (\pm k + \frac{1}{2})\pi)$, $k$ a positive integer, and no nonzero zero in $(-\frac{\pi}{2a}, \frac{\pi}{2a})$. The functions $\tan(x)$ and $\tanh(x)$ for $x \in \mathbb{R}$ are odd functions, so the function $\tan(x) - \tanh(x)$, $x \in \mathbb{R}$ is an odd function. Thus $\hat{\mu}_k^- = -\hat{\mu}_k^+$, and since $\tanh(x) \to 1$ as $x \to \infty$, we have

$$\hat{\mu}_k^+ = (4k-3)\frac{\pi}{4a} + o(1),\ \hat{\mu}_k^- = -(4k-3)\frac{\pi}{4a} + o(1),\ k = 2, 3, 4, \ldots. \tag{4.123}$$

As $\tan(i\gamma a) = i\tanh(\gamma a)$ and $\tanh(i\gamma a) = i\tan(\gamma a)$, the nonzero pure imaginary zeros of $\phi_0$ are simple and of the form

$$\hat{\mu}_{-k}^+ = i(4k-3)\frac{\pi}{4a} + o(1),\ \hat{\mu}_{-k}^- = -i(4k-3)\frac{\pi}{4a} + o(1),\ k = 2, 3, 4, \ldots. \tag{4.124}$$

**Remark 4.26.** We know that

$$\sin(\mu a)\cosh(\mu a) = \frac{1}{2}\left[\sin((1+i)\mu a) + \sin((1-i)\mu a)\right], \tag{4.125}$$

$$\cos(\mu a)\sinh(\mu a) = \frac{1}{2}\left[-i\sin((1+i)\mu a) + i\sin((1-i)\mu a)\right]. \tag{4.126}$$

Thus

$$\begin{aligned}\phi_0(\mu) &= \mu^3(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \\ &= \frac{1}{2}\mu^3\left(\sin((1+i)\mu a) + \sin((1-i)\mu a) + i\sin((1+i)\mu a) - i\sin((1-i)\mu a)\right) \\ &= \frac{1}{2}\mu^3((1+i)\sin((1+i)\mu a) + (1-i)\sin((1-i)\mu a)).\end{aligned} \tag{4.127}$$

Putting $(1+i)\mu a = x+iy$, $x, y \in \mathbb{R}$, it follows for $\mu \neq 0$ that

$$
\begin{aligned}
\phi_0(\mu) = 0 \Rightarrow & |(1+i)\sin((1+i)\mu a)| = |(1-i)\sin((1-i)\mu a)| \\
& \Leftrightarrow |\sin((1+i)\mu a)| = |\sin((1-i)\mu a)| \\
& \Leftrightarrow |\sin(x+iy)| = |\sin(y-ix)| \\
& \Leftrightarrow \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 y \cosh^2 x + \cos^2 y \sinh^2 x \\
& \Leftrightarrow \sin^2 x \cosh^2 y + \cos^2 x(\cosh^2 y - 1) = \sin^2 y \cosh^2 x + \cos^2 y(\cosh^2 x - 1) \\
& \Leftrightarrow \cosh^2 y - \cos^2 x = \cosh^2 x - \cos^2 y \\
& \Leftrightarrow \cosh^2(|y|) + \cos^2(|y|) = \cosh^2(|x|) + \cos^2(|x|). \qquad (4.128)
\end{aligned}
$$

The function $\psi : x \mapsto \cosh^2 x + \cos^2 x = \frac{1}{2}\cosh(2x) + \frac{1}{2}\cos(2x) + 1$ is even and has a positive derivative on $(0, \infty)$, thus the function $\psi$ is strictly increasing $[0, \infty)$. Hence it follows from (4.128) that $\phi_0(\mu) = 0$ implies that $|y| = |x|$ and therefore $y = \pm x$. Whence

$$\mu = \frac{x+iy}{(1+i)a} = \frac{1 \pm i}{1+i}\,\frac{x}{a}$$

is either real or pure imaginary. Hence all the zeros of $\phi_0$ are real numbers or pure imaginary numbers.

Therefore the zeros of $\phi_0$ are 0 and the zeros given in the forms (4.123) and (4.124), which are more suitable for their enumeration with their multiplicity.

Let $\mu \mapsto \psi_0(\mu) = \mu^3$ and

$$\psi_3(\mu) = \tan(\mu a) - \tanh(\mu a). \qquad (4.129)$$

We have

$$
\begin{cases}
\psi_3'(\mu) = a(1+\tan^2(\mu a)) - \frac{a}{\cosh^2(\mu a)}, \\
\psi_3''(\mu a) = 2a^2\left[(1+\tan^2(\mu a))\tan(\mu a) + \frac{\tanh(\mu a)}{\cosh^2(\mu a)}\right], \\
\psi_3^{(3)}(\mu a) = 2a^3\left[2(1+\tan^2(\mu a))\tan^2(\mu a) + (1+\tan^2(\mu a))^2 + \frac{1-2\sinh^2(\mu a)}{\cosh^4(\mu a)}\right].
\end{cases} \qquad (4.130)
$$

It is easy to check that 0 is a zero of multiplicity 3 of $\psi_0$. On the other hand we have $\psi_3(0) = \psi_3'(0) = \psi_3''(0) = 0$, while $\psi_3^{(3)}(0) = 4a^3 \neq 0$. Thus 0 is a zero of multiplicity 3 of $\psi_3$, therefore 0 is a zero of multiplicity 6 of $\phi_0$. We recall that $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$, $k = 2, 3, \ldots$, are simple zeros of $\phi_0$. Hence it follows from (4.123) and (4.124) that the zeros of $\phi_0$ counted with multiplicity are

$$\left.\begin{array}{l}\hat{\mu}^{\pm}_{-1} = 0,\ \hat{\mu}^{\pm}_0 = 0,\ \hat{\mu}^{\pm}_1 = 0,\ \hat{\mu}^{\pm}_k = \pm(4k-3)\frac{\pi}{4a} + o(1),\\ \hat{\mu}^{\pm}_{-k} = \pm i(4k-3)\frac{\pi}{4a} + o(1),\ k = 2, 3, \ldots\end{array}\right\}, \tag{4.131}$$

see (4.123) and (4.124).

Let

$$\phi_{00}(\mu) = -\cos(\mu a) + \sin(\mu a), \tag{4.132}$$

$$\phi_{01}(\mu) = (1 - \tanh(\mu a))\cos(\mu a) - \frac{2i\alpha}{\mu}\sin(\mu a)\tanh(\mu a) - \frac{2i\cos(\mu a)}{\alpha\mu^3} + \frac{1}{\mu^4}(\sin(\mu a) - \cos(\mu a)\tanh(\mu a)). \tag{4.133}$$

Then

$$\phi_{02}(\mu) := \frac{\phi(\mu)}{\alpha\mu^3\cosh(\mu a)} = \phi_{00}(\mu) + \phi_{01}(\mu). \tag{4.134}$$

it easy to check that

$$\mu^{00}_k = \left(\frac{\pi}{4a} + k\frac{\pi}{a}\right), \quad k \in \mathbb{Z},\ \text{are the zeros of } \phi_{00}, \tag{4.135}$$

and that

$$\widetilde{\mu}^{00}_k = i\left(\frac{\pi}{4a} + k\frac{\pi}{a}\right),\ k \in \mathbb{Z},\ \text{are the images of } \mu^{00}_k \text{ by the rotation of angle } \frac{\pi}{2}. \tag{4.136}$$

Let

$$\left.\begin{array}{l}R_k \text{ be the rectangles with vertices } (4k+1)\frac{\pi}{4a} \pm \varepsilon \pm i\varepsilon,\ k \in \mathbb{Z} \text{ and}\\ \varepsilon \in (0, \frac{\pi}{2a}),\ R_{-k} \text{ its symmetric image with the } y \text{ axis}, \widetilde{R}_k \text{ and } \widetilde{R}_{-k} \text{ the}\\ \text{respective images of the rectangles } R_k \text{ and } R_{-k} \text{ by the rotation of angle } \frac{\pi}{2}.\end{array}\right\}. \tag{4.137}$$

**Remark 4.27.** We can observe that the rectangles $R_k$ contain the zeros $\mu^{00+}_k = \left(\frac{\pi}{4a} + k\frac{\pi}{a}\right)$, thus $-\mu^{00+}_k = -\left(\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in R_{-k}$, $\widetilde{\mu}^{00+}_k = i\left(\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_k$, while $-\widetilde{\mu}^{00+}_k = -i\left(\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_{-k}$, where $k \in \mathbb{Z}$.

Due to $\varepsilon < \frac{\pi}{2a}$, the rectangles $R_k$, $k \in \mathbb{Z}$ do not intersect, as well as the rectangles $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$. Since $|\phi_{00}|$ is periodic of period $\frac{\pi}{a}$, there exists a constant $\rho(\varepsilon) > 0$ such that

$|\phi_{00}(\mu)| > \rho(\varepsilon)$ for all $\mu$ on the rectangle $R_k$, $k \in \mathbb{Z}$. Let $\mu a = x + iy$. Then for sufficiently large positive $|k| \geq k_0(\varepsilon)$, we have

$$\left.\begin{aligned} &|\cos(\mu a)(-1+\tanh(\mu a)| = \left|\tfrac{e^{(y-ix)}+e^{-(y-ix)}}{e^{2\mu a}+1}\right| \leq \tfrac{2e^{\Im|\mu a|}}{e^{2|\Re\mu a|}} < 3e^{-|\Re\mu a|} \\ &|\cos(\mu a)| < \sqrt{2}, \quad |\sin(\mu a)| < \sqrt{2}, \quad |\tanh(\mu a)| < 2 \\ &|\sin(\mu a) - \cos(\mu a)\tanh(\mu a)| < 3\sqrt{2} \end{aligned}\right\} \tag{4.138}$$

for all $\mu$ on the rectangle $R_k$, where $|k|$ is large enough. Thus, by using (4.138), we obtain for all $\mu$ on the rectangle $R_k$ and $|k| \geq k_0(\varepsilon)$ large enough,

$$|\phi_{01}(\mu)| < \frac{2\alpha\sqrt{2}}{|\mu|} + \frac{2\sqrt{2}}{\alpha|\mu|^3} + \frac{3\sqrt{2}}{|\mu|^4} + 3e^{-|\Re\mu a|}. \tag{4.139}$$

Since the right hand side tends to 0 as $|\Re\mu a| \to \infty$, it follows that for all $\mu$ on the rectangle $R_k$, where $k \in \mathbb{Z}$, $|k| > k_0(\varepsilon)$,

$$|\phi_{01}(\mu)| < |\phi_{00}(\mu)|. \tag{4.140}$$

Since $\phi$ is an even function, see (4.120), (4.121) and (4.122), we have the same estimates (4.139) and (4.140) for all $\mu$ on the rectangles $R_{-k}$ where $|k| \geq k_0(\varepsilon)$ large enough.

Let

$$\begin{aligned} \widetilde{\phi}_0(\mu) &= \alpha\mu^3(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \\ &\quad + \frac{\alpha}{\mu}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \end{aligned} \tag{4.141}$$

and

$$\widetilde{\phi}_1(\mu) = -2i\alpha^2\mu^2\sin(\mu a)\sinh(\mu a) - 2i\cos(\mu a)\cosh(\mu a). \tag{4.142}$$

Thus

$$\phi(\mu) = \widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu). \tag{4.143}$$

For all $\mu \in \mathbb{C}$, we have

$$\begin{aligned}\widetilde{\phi}_0(i\mu) &= \alpha(i\mu)^3(\sin(i\mu a)\cosh(i\mu a) - \cos(i\mu a)\sinh(i\mu a)) \\ &+ \frac{\alpha}{i\mu}(\sin(i\mu a)\cosh(i\mu a) - \cos(i\mu a)\sinh(i\mu a)) \\ &= -i\alpha\mu^3(i\sinh(\mu a)\cos(\mu a) - i\sin(\mu a)\cosh(\mu a)) \\ &+ \frac{\alpha}{i\mu}(i\sinh(\mu a)\cos(\mu a) - i\cosh(\mu a)\sin(\mu a)) \\ &= -i\alpha\mu^3(-i)(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \\ &- \frac{i\alpha}{i\mu}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) = -\widetilde{\phi}_0(\mu),\end{aligned} \tag{4.144}$$

while

$$\begin{aligned}\widetilde{\phi}_1(i\mu) &= -2i\alpha^2(i\mu)^2\sin(i\mu a)\sinh(i\mu a) - 2i\cos(i\mu a)\cosh(i\mu a). \\ &= -2i\alpha^2\mu^2\sin(\mu a)\sinh(\mu a) - 2i\cos(\mu a)\cosh(\mu a) = \widetilde{\phi}_1(\mu).\end{aligned} \tag{4.145}$$

Hence for all $\mu \in \mathbb{C}$

$$\phi(i\mu) = -\widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu). \tag{4.146}$$

By using the equations (4.132), (4.133) and (4.134) we have obtained the estimates (4.139) and (4.140) for all $\mu$ on the rectangles $R_k$ and $R_{-k}$, since $\phi$ is an even function. On the other hand, it follows from (4.143) and (4.146) that $|\phi(\mu)|$ and $|\phi(i\mu)|$ have the same upper bound $|\widetilde{\phi}_0(\mu)| + |\widetilde{\phi}_1(\mu)|$ for all $\mu$ on the rectangle $R_k$ or $R_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough. Since we can obtain the estimates (4.139) and (4.140) for all $\mu$ on the rectangle $R_k$ or $R_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough, then we can also obtain the same estimates for all $\mu$ on the rectangle $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough. As the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{Z}$ are closed curves, 0 is a zero of multiplicity 6, $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$ are simple zeros of $\phi_0$ and $\phi_0$ has no nonzero zero in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, then (4.132), (4.133), (4.134), (4.140), Remark 4.27 and Rouché's theorem imply that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$ and the images of these zeros by the rotation of angle $\frac{\pi}{2}$, where these asymptotics are

$$\left.\begin{aligned}\hat{\mu}_k^{\pm} &= \pm(4k-3)\tfrac{\pi}{4a} + o(1) \text{ with } k \geq k_0(\varepsilon) \text{ and} \\ \hat{\mu}_k^{\pm} &= \pm i(4|k|-3)\tfrac{\pi}{4a} + o(1) \text{ with } k \leq -k_0(\varepsilon)\end{aligned}\right\}, \tag{4.147}$$

where $o(1) \to 0$ as $|k| \to \infty$, see (4.131).

Let $S_k$, $k \in \mathbb{N}$ be the square defined in (4.66). Let

$$\phi_{11}(\mu) = \frac{\alpha}{\mu}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)), \tag{4.148}$$

$$\phi_{12}(\mu) = -2i\alpha^2\mu^2 \sin(\mu a)\sinh(\mu a), \tag{4.149}$$

$$\phi_{13}(\mu) = -2i\cos(\mu a)\cosh(\mu a). \tag{4.150}$$

Then there exists $k_0 = \frac{a}{\pi}\sqrt[4]{4}$ such that for all $\mu$ on the square $S_k$, $k \in \mathbb{N}$ and $k \geq k_0$, $\frac{1}{|\mu|} < \frac{|\mu|^3}{3}$, thus for $k \in \mathbb{N}$ and $k \geq k_0$,

$$|\phi_{11}(\mu)| < \frac{\alpha}{3}|\phi_0(\mu)|. \tag{4.151}$$

Let

$$\phi_2(\mu) = \coth(\mu a) - \cot(\mu a). \tag{4.152}$$

Since

$$\cot\left(\left(\frac{k\pi}{a} + i\gamma\right)a\right) = \cot(i\gamma a) = -i\coth(\gamma a) \in i\mathbb{R} \cup \{\infty\}, \tag{4.153}$$

then for $k \in \mathbb{Z}$ and for $\mu = \frac{k\pi}{a} + i\gamma$, where $\gamma \in \mathbb{R}$, we have

$$|\cot(\mu a) \pm 1| \geq 1. \tag{4.154}$$

Since for $\mu \neq 0$, $\coth(\mu a) = \frac{1}{\tanh(\mu a)}$, then it follows from (4.49) that we have for $\mu = x + iy$, where $x, y \in \mathbb{R}$ and $x \neq 0$,

$$\coth(\mu a) \to \pm 1, \tag{4.155}$$

uniformly in $y$ as $x \to \pm\infty$. Thus there exists $\widetilde{k}_0 \in \mathbb{N}$ such that for $k \in \mathbb{Z}$ and $|k| \geq \widetilde{k}_0$,

$$\left|\coth\left(\left(\frac{k\pi}{a} + i\gamma\right)a\right) - \operatorname{sgn}(k)\right| < \frac{1}{2}. \tag{4.156}$$

It follows from (4.154) and (4.156) that for $\mu = \frac{k\pi}{a} + i\gamma$, $k \geq \widetilde{k}_0$, $\gamma \in \mathbb{R}$

$$|\phi_2(\mu)| \geq \frac{1}{2}. \tag{4.157}$$

By interchanging coth and cot we obtain for $\mu = \gamma + i\frac{k\pi}{a}$, $k \in \mathbb{Z}$, $|k| \geq \widetilde{k}_0$ and $\gamma \in \mathbb{R}$,

$$\begin{aligned}\coth\left(\left(\gamma + ik\frac{\pi}{a}\right)a\right) &= \coth(\gamma a + ik\pi) = \coth(i(-i\gamma a + k\pi)) \\ &= -i\cot(-i\gamma a + k\pi) = i\coth(i\gamma a) = \cot(\gamma a).\end{aligned} \tag{4.158}$$

It follows from (4.153), (4.154), (4.156) and (4.158) that for $\mu = \gamma + i\frac{k\pi}{a}$, $k \in \mathbb{Z}$, $|k| \geq \widetilde{k}_0$ and $\gamma \in \mathbb{R}$ that

$$|\coth(\mu a) \pm 1| \geq 1 \tag{4.159}$$

while

$$|\cot(\mu a) - \operatorname{sgn}(k)| < \frac{1}{2}. \tag{4.160}$$

Thus, for $\mu = \gamma + ik\frac{\pi}{a}$ $k \in \mathbb{Z}$, $|k| \geq \widetilde{k}_0$ and $\gamma \in \mathbb{R}$,

$$|\phi_2(\mu)| \geq \frac{1}{2}. \tag{4.161}$$

Hence for all $\mu$ on the square $S_k$, where $k \geq \widetilde{k}_0$,

$$|\phi_2(\mu)| \geq \frac{1}{2}. \tag{4.162}$$

Whence for all $\mu$ on the square with the vertices $\pm k\frac{\pi}{a} \pm ik\frac{\pi}{a}$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_0 = \max\{\widetilde{k}_0, \frac{13\alpha a}{\pi}\}$

$$\begin{aligned}
\frac{\alpha}{3}|\phi_0(\mu)| &= \frac{\alpha}{3}|\mu|^3|\phi_2(\mu)||\sin(\mu a)\sinh(\mu a)| \\
&\geq \frac{\alpha}{6}|\mu|^3|\sin(\mu a)\sinh(\mu a)| \\
&= \frac{|\mu|}{12\alpha}2\alpha^2|\mu|^2|\sin(\mu a)\sinh(\mu a)| \\
&= \frac{|\mu|}{12\alpha}|\phi_{12}(\mu)| > |\phi_{12}(\mu)|. \qquad (4.163)
\end{aligned}$$

Let

$$\phi_3(\mu) = \tan(\mu a) - \tanh(\mu a). \tag{4.164}$$

Since $\tan((\frac{k\pi}{a} + i\gamma)a) = \tan(i\gamma a) = i\tanh(\gamma a) \in i\mathbb{R} \cup \{\infty\}$, then for $k \in \mathbb{Z}$ and $\mu = \frac{k\pi}{a} + i\gamma$, where $\gamma \in \mathbb{R}$, we have

$$|\tan(\mu a) \pm 1| \geq 1. \tag{4.165}$$

For $\mu = x + iy$, $x, y \in \mathbb{R}$ and $x \neq 0$, we have $\tanh(\mu a) \to \pm 1$ uniformly in $y$ as $x \to \pm\infty$, see (4.49). Hence there exists $\hat{k}_0 \in \mathbb{N}$ such that for all $k \in \mathbb{Z}$, $|k| \geq \hat{k}_0$ and $\gamma \in \mathbb{R}$,

$$\left|\tanh\left(\left(\frac{k\pi}{a} + i\gamma\right)a\right) - \operatorname{sgn}(k)\right| < \frac{1}{2}. \tag{4.166}$$

It follows from (4.165) and (4.166) that

$$|\phi_3(\mu)| \geq \frac{1}{2} \text{ for } \mu = \frac{k\pi}{a} + i\gamma, k \in \mathbb{Z}, |k| \geq \hat{k}_0 \text{ and } \gamma \in \mathbb{R}. \tag{4.167}$$

By interchanging tan and tanh we obtain the same estimate for $\mu = \gamma + i\frac{k\pi}{a}$, $k \in \mathbb{Z}$, $k \geq \hat{k}_0$ and $\gamma \in \mathbb{R}$. Thus for $\mu = i\frac{k\pi}{a} + \gamma$, $k \in \mathbb{Z}$, $|k| \geq \hat{k}_0$ and $\gamma \in \mathbb{R}$,

$$|\phi_3(\mu)| \geq \frac{1}{2}, \tag{4.168}$$

see (4.165), (4.166), and (4.167). Hence (4.167) and (4.168) imply that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$, $k \geq \hat{k}_0$

$$|\phi_3(\mu)| \geq \frac{1}{2}. \tag{4.169}$$

Therefore for all $\mu$ on the square with the vertices $\pm k\frac{\pi}{a} \pm ik\frac{\pi}{a}$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_0 = \max\left\{\hat{k}_0, \frac{a}{\pi}\sqrt[3]{\frac{13}{\alpha}}\right\}$

$$\begin{aligned}
\frac{\alpha}{3}|\phi_0(\mu)| &= \frac{\alpha}{3}|\mu|^3|\phi_3(\mu)||\cos(\mu a)\cosh(\mu a)| \\
&\geq \frac{\alpha}{6}|\mu|^3|\cos(\mu a)\cosh(\mu a)| \\
&= \frac{|\mu|^3\alpha}{12}2|\cos(\mu a)\cosh(\mu a)| \\
&= \frac{\alpha|\mu|^3}{12}|\phi_{13}(\mu)| > |\phi_{13}(\mu)|.
\end{aligned} \tag{4.170}$$

Let $\bar{m}_0 = \max\{k_0, \widetilde{m}_0, \hat{m}_0\}$. Then it follows from (4.151), (4.163) and (4.170) that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \bar{m}_0$,

$$|\phi_1(\mu)| < \alpha|\phi_0(\mu)|. \tag{4.171}$$

Since the square $S_k$ is a closed curve, then (4.171) and Rouché's theorem imply that $\phi$ and $\phi_0$ have the same number of zeros inside the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \bar{m}_0$.

**Remark 4.28.** Let $k_1 = \max\{k_0(\varepsilon), \bar{m}_0\}$. Since 0 is a zero of multiplicity 6 of $\phi_0$, while $\hat{\mu}_k^+$, $\hat{\mu}_k^-$, $\hat{\mu}_{-k}^+$ and $\hat{\mu}_{-k}^-$ are its simple zeros, then the number of zeros of $\phi_0$ and therefore of $\phi$ inside the squares $S_k$, where $k \geq k_1$ is $4k+2$. Thus we have the following proposition.

**Proposition 4.29.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2), *where $B_1(y) = y(0)$, $B_2(y) = y''(0)$, $B_3 y = y'(a) - i\alpha\lambda y''(a)$ and $B_4 y = y^{(3)}(a) - i\alpha\lambda y(a)$ , counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{cases}
\hat{\mu}_k^{\pm} &= \pm(4k-3)\frac{\pi}{4a} + o(1), \quad if\ k > 0, \\
\hat{\mu}_k^{\pm} &= \pm i(4|k|-3)\frac{\pi}{4a} + o(1), \quad if\ k < 0.
\end{cases}$$

*In particular, there is an odd number of pure imaginary eigenvalues.*

**Remark 4.30.** We give the enumeration of the zeros of $\phi$ inside the square $S_{k_1}$. The remainder of the proof of the above proposition is identical to the remainder of the proof of Proposition 4.23.

*Proof.* The function $\phi$ has $4k_1+2$ number of zeros inside the square $S_{k_1}$. Since the function $\phi_0$ has no nonzero zeros in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, then the zeros of $\phi$ inside the square $S_{k_1}$ are the following

$$\hat{\mu}_0^{\pm} = \pm\left(\frac{-3\pi}{4a}\right) + o(1),\ \hat{\mu}_1^{\pm} = \pm\left(\frac{\pi}{4a}\right) + o(1), \dots, \hat{\mu}_{k_1}^{\pm} = \pm(4k_1-3)\frac{\pi}{4a} + o(1),$$
$$\hat{\mu}_{-1}^{\pm} = \pm i\left(\frac{\pi}{4a}\right) + o(1),\ \hat{\mu}_{-2}^{\pm} = \pm i\left(\frac{5\pi}{4a}\right) + o(1), \dots, \hat{\mu}_{-(k_1)}^{\pm} = \pm i(4k_1-3)\frac{\pi}{4a} + o(1).$$

Hence

$$\begin{aligned} \hat{\mu}_k^{\pm} &= \pm(4k-3)\frac{\pi}{4a} + o(1),\ \text{where } 0 \le k \le k_1, \\ \hat{\mu}_k^{\pm} &= \pm i(4|k|-3)\frac{\pi}{4a} + o(1),\ \text{where } -k_1 \le k \le -1. \end{aligned}$$

Using the same approach as in the proof of Proposition 4.23, we can prove that the zeros of $\phi$ for $|k| \ge k_1$ are

$$\begin{aligned} \hat{\mu}_k^{\pm},\ k &= k_1, k_1+1, \dots \\ &-k_1, -k_1-1, \dots \end{aligned}$$

and satisfy

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(4k-3)\frac{\pi}{4a} + o(1),\ \text{where } k \ge k_1, \\ \hat{\mu}_k^{\pm} = \pm i(4|k|-3)\frac{\pi}{4a} + o(1),\ \text{where } k \le -k_1, \end{cases}$$

see (4.147). $\square$

### 4.4.4 Asymptotic of the eigenvalues for $B_3y = y'(a) - i\alpha\lambda y''(a)$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a)$

We have from (4.23),

$$\begin{aligned}
\det M &= B_3y_2B_4y_4 - B_4y_2B_3y_4 \\
&= (y_2'(a) - i\alpha\mu^2 y_2''(a))(y_4(a) + i\alpha\mu^2 y_4^{(3)}(a)) - (y_2(a) + i\alpha\mu^2 y_2^{(3)}(a))(y_4'(a) - i\alpha\mu^2 y_4''(a)) \\
&= y_2'(a)y_4(a) + \alpha^2\mu^4 y_4^{(3)}(a)y_2''(a) - y_4'(a)y_2(a) - \alpha^2\mu^4 y_4''(a)y_2^{(3)}(a) \\
&- i\alpha\mu^2(y_4(a)y_2''(a) - y_4^{(3)}(a)y_2'(a) + y_4'(a)y_2^{(3)}(a) - y_4''(a)y_2(a)).
\end{aligned} \tag{4.172}$$

Il follows from (4.26) that

$$\begin{aligned}
\det M &= -y_4''(a)y_4'(a) - \alpha^2\mu^8 y_4'(a)y_4''(a) + y_4^{(3)}(a)y_4(a) + \alpha^2\mu^8 y_4(a)y_4^{(3)}(a) \\
&+ i\alpha\mu^2((y_4''(a))^2 - \mu^4(y_4'(a))^2 + (y_4^{(3)}(a))^2 - \mu^4(y_4(a))^2) \\
&= -(\alpha^2\mu^8 + 1)(y_4''(a)y_4'(a) - y_4(a)y_4^{(3)}(a)) \\
&+ i\alpha\mu^2((y_4''(a))^2 - \mu^4(y_4'(a))^2 + (y_4^{(3)}(a))^2 - \mu^4(y_4(a))^2).
\end{aligned} \tag{4.173}$$

However

$$y_4''(a)y_4'(a) - y_4(a)y_4^{(3)}(a) = \frac{1}{2\mu^3}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \tag{4.174}$$

see (4.34). Let

$$A(a) = (y_4''(a))^2 - \mu^4(y_4'(a))^2 + (y_4^{(3)}(a))^2 - \mu^4(y_4(a))^2. \tag{4.175}$$

Then (4.30), (4.31), (4.32) and (4.33) give

$$\begin{aligned}
A(a) &= \frac{1}{4\mu^2}\sin^2(\mu a) + \frac{1}{2\mu^2}\sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^2}\sinh^2(\mu a) \\
&- \mu^4\left(\frac{1}{4\mu^4}\cos^2(\mu a) - \frac{1}{2\mu^4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4\mu^4}\cosh^2(\mu a)\right) \\
&+ \frac{1}{4}\cos^2(\mu a) + \frac{1}{2}\cos(\mu a)\cosh(\mu a) + \frac{1}{4}\cosh^2(\mu a) \\
&- \mu^4\left(\frac{1}{4\mu^6}\sin^2(\mu a) - \frac{1}{2\mu^6}\sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^6}\sinh^2(\mu a)\right) \\
&= \frac{1}{\mu^2}\sin(\mu a)\sinh(\mu a) + \cos(\mu a)\cosh(\mu a).
\end{aligned} \tag{4.176}$$

It follows from (4.173), (4.174) and (4.176) that

$$\begin{aligned}\det M = {} & \frac{\alpha^2\mu^5}{2}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) \\ & + i\alpha\mu^2\cos(\mu a)\cosh(\mu a) + i\alpha\sin(\mu a)\sinh(\mu a) \\ & + \frac{1}{2\mu^3}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)). \end{aligned} \tag{4.177}$$

Hence the characteristic equation $2\det M = 0$ is

$$\phi(\mu) = \alpha^2\phi_0(\mu) + \phi_1(\mu) = 0, \tag{4.178}$$

where

$$\phi_0(\mu) = \mu^5(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) \tag{4.179}$$

$$\begin{aligned}\phi_1(\mu) = {} & 2i\alpha\mu^2\cos(\mu a)\cosh(\mu a) + 2i\alpha\sin(\mu a)\sinh(\mu a) \\ & + \frac{1}{\mu^3}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)). \end{aligned} \tag{4.180}$$

The function $\phi_0$ has the same zeros as the function $\phi_0$ defined in Subsection 4.4.3, see (4.118) and (4.179). However 0 is a zero of multiplicity 6 for the function $\phi_0$ defined in (4.118), while it is a zero of multiplicity 8 for the function $\phi_0$ defined in (4.179). Hence we use in this subsection the following form of the zeros of $\phi_0$ more suitable for their enumeration with their multiplicity.

$$0,\ \hat{\mu}_k^{\pm} = \pm(4k-7)\frac{\pi}{4a} + o(1),\ \hat{\mu}_{-k}^{\pm} = \pm i(4k-7)\frac{\pi}{4a} + o(1),\ k = 3, 4, \ldots. \tag{4.181}$$

We recall that 0 is a zero of multiplicity 3 of $\psi(\mu) = \cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)$, see (4.129) and (4.130), since 0 is a zero of multiplicity of $\psi_0(\mu) = \mu^5$, then 0 is a zero of multiplicity 8 of $\phi_0$, while $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$, $k = 1, 2, \ldots$ are its simple zeros. All the zeros of $\phi_0$ are either real or pure imaginary, see Remark 4.26. Thus, it follows from (4.181) that, the zeros of $\phi_0$ counted with multiplicity are

$$\left.\begin{array}{l}\hat{\mu}_{-2}^{\pm} = 0,\ \hat{\mu}_{-1}^{\pm} = 0,\ \hat{\mu}_1^{\pm} = 0,\ \hat{\mu}_2^{\pm} = 0,\ \hat{\mu}_k^{\pm} = \pm(4k-7)\frac{\pi}{4a} + o(1), \\ \hat{\mu}_{-k}^{\pm} = \pm i(4k-7)\frac{\pi}{4a} + o(1),\ k = 3, 4, \ldots \end{array}\right\}. \tag{4.182}$$

Let

$$\phi_{00}(\mu) = \cos(\mu a) - \sin(\mu a) \tag{4.183}$$

$$\begin{aligned}\phi_{01}(\mu) &= \cos(\mu a)(1 - \tanh(\mu a)) + \frac{2i}{\alpha\mu^3}\cos(\mu a) + \frac{2i}{\alpha\mu^5}\sin(\mu a)\tanh(\mu a) \\ &\quad + \frac{1}{\alpha^2\mu^8}(\cos(\mu a)\tanh(\mu a) - \sin(\mu a)).\end{aligned} \tag{4.184}$$

Then

$$\phi_{02}(\mu) := \frac{\phi(\mu)}{\alpha^2\mu^5\cosh(\mu a)} = \phi_{00}(\mu) + \phi_{01}(\mu). \tag{4.185}$$

We recall that

$$\mu_k^{00} = \left(\frac{\pi}{4a} + k\frac{\pi}{a}\right),\ k \in \mathbb{Z},$$

are the zeros of $\phi_{00}$, see (4.132) and (4.183), while

$$\widetilde{\mu}_k^{00} = i\left(\frac{\pi}{4a} + k\frac{\pi}{a}\right),\ k \in \mathbb{Z},$$

are the images of $\mu_k^{00}$ by the rotation of angle $\frac{\pi}{2}$. Let $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{Z}$, be the rectangles defined in (4.137). We recall that the rectangles $R_k$, $k \in \mathbb{Z}$, do not intersect, as well as the rectangles $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ due to $\varepsilon < \frac{\pi}{2a}$, we recall, also, that as $|\phi_{00}|$ is periodic of period $\frac{\pi}{a}$, then there exists a constant $\rho(\varepsilon) > 0$ such that $|\phi_{00}(\mu)| > \rho(\varepsilon)$ for all $\mu$ on the rectangle $R_k$, $k \in \mathbb{Z}$. Thus it follows from (4.138) that for all $\mu$ on the rectangle $R_k$, where $|k| \geq k_0(\varepsilon)$ large enough,

$$|\phi_{01}(\mu)| < \frac{2\sqrt{2}}{\alpha|\mu|^3} + \frac{2\sqrt{2}}{\alpha|\mu|^5} + \frac{3\sqrt{2}}{\alpha^2|\mu|^8} + 3e^{-|\Re\mu a|}. \tag{4.186}$$

Since the right hand tends to 0 as $|\Re\mu a| \to \infty$, then for all $\mu$ on the rectangle $R_k$, where $|k| \geq k_0(\varepsilon)$,

$$|\phi_{01}(\mu)| < |\phi_{00}(\mu)|. \tag{4.187}$$

For all $\mu \in \mathbb{C}$, we have

$$\begin{aligned}\phi_0(-\mu) &= (-\mu)^5(\cos(-\mu a)\sinh(-\mu a) - \sin(-\mu a)\cosh(-\mu a)) \\ &= -\mu^5(-\cos(\mu a)\sinh(\mu a) + \sin(\mu a)\cosh(\mu a)) \\ &= \mu^5(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) = \phi_0(\mu),\end{aligned} \tag{4.188}$$

while

$$\begin{aligned}
\phi_1(-\mu) &= 2i\alpha(-\mu)^2\cos(-\mu a)\cosh(-\mu a) + 2i\alpha\sin(-\mu a)\sinh(-\mu a) \\
&+ \frac{1}{(-\mu)^3}(\cos(-\mu a)\sinh(-\mu a) - \sin(-\mu a)\cosh-(\mu a)) \\
&= 2i\alpha\mu^2\cos(\mu a)\cosh(\mu a) + 2i\alpha\sin(\mu a)\sinh(\mu a) \\
&+ \frac{1}{\mu^3}(-\cos(\mu a)\sinh(\mu a) + \sin(\mu a)\cosh(\mu a)) \\
&= 2i\alpha\mu^2\cos(\mu a)\cosh(\mu a) + 2i\alpha\sin(\mu a)\sinh(\mu a) \\
&+ \frac{1}{\mu^3}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) = \phi_1(\mu).
\end{aligned} \tag{4.189}$$

It follows from (4.188) and (4.189) that $\phi$ is an even function. Thus we have the same estimates (4.186) and (4.187) for all $\mu$ on the rectangles $R_k$ and $R_{-k}$ where $k \in \mathbb{N}$ and $k \geq k_0(\varepsilon)$ large enough.

Let

$$\widetilde{\phi}_0(\mu) = \alpha^2\mu^5(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) \tag{4.190}$$

and

$$\begin{aligned}
\widetilde{\phi}_1(\mu) &= -2i\alpha\sin(\mu a)\sinh(\mu a) - 2i\alpha\mu^2\cos(\mu a)\cosh(\mu a) \\
&- \frac{1}{\mu^3}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)).
\end{aligned} \tag{4.191}$$

Thus

$$\phi(\mu) = \widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu). \tag{4.192}$$

For all $\mu \in \mathbb{C}$, we have

$$\begin{aligned}
\widetilde{\phi}_0(i\mu) &= \alpha^2(i\mu)^5(\cos(i\mu a)\sinh(i\mu a) - \sin(i\mu a)\cosh(i\mu a)) \\
&= \alpha^2 i\mu^5(i\cosh(\mu a)\sin(\mu a) - i\sinh(\mu a)\cos(\mu a)) \\
&= \alpha^2\mu^5(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) = \widetilde{\phi}_0(\mu),
\end{aligned} \tag{4.193}$$

while

$$\begin{aligned}\widetilde{\phi}_1(i\mu) &= 2i\alpha\sin(i\mu a)\sinh(i\mu a) + 2i\alpha(i\mu)^2\cos(i\mu a)\cosh(i\mu a)\\ &+ \frac{1}{(i\mu)^3}(\cos(i\mu a)\sinh(i\mu a) - \sin(i\mu a)\cosh(i\mu a))\\ &= -2i\alpha\sin(\mu a)\sinh(\mu a) - 2i\alpha\mu^2\cos(\mu a)\cosh(\mu a)\\ &+ \frac{1}{i\mu^3}(i\cosh(\mu a)\sin(\mu a) - i\sinh(\mu a)\cos(\mu a))\\ &= -2i\alpha\sin(\mu a)\sinh(\mu a) - 2i\alpha\mu^2\cos(\mu a)\cosh(\mu a)\\ &- \frac{1}{\mu^3}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) = -\widetilde{\phi}_1(\mu).\end{aligned} \tag{4.194}$$

Thus

$$\widetilde{\phi}_0(i\mu) = \widetilde{\phi}_0(\mu), \tag{4.195}$$

while

$$\widetilde{\phi}_1(i\mu) = -\widetilde{\phi}_1(\mu). \tag{4.196}$$

Whence

$$\phi(i\mu) = \widetilde{\phi}_0(\mu) - \widetilde{\phi}_1(\mu). \tag{4.197}$$

It follows from (4.183), (4.184) and (4.185) that we can obtain the same estimates (4.186) and (4.187) for all $\mu$ on the rectangles $R_k$ and $R_{-k}$ where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough. Since it follows from (4.192) and (4.197) that $\phi(\mu)$ and $\phi(i\mu)$ have the same upper bound $|\widetilde{\phi}_0(\mu)| + |\widetilde{\phi}_1(\mu)|$ for all $\mu$ on the rectangles $R_k$ and $R_{-k}$, then we can obtain the same estimates (4.186) and (4.187) for all $\mu$ on the $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough.

We recall that $\mu_k^{00+} = \left(\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in R_k$, $-\mu_k^{00+} = -\left(\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in R_{-k}$, $\widetilde{\mu}_k^{00+} = i\left(\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_k$ and $-\widetilde{\mu}_k^{00+} = -i\left(\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_{-k}$, see Remark 4.27. As $\phi_0$ has no nonzero zero in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{Z}$ are closed curves, 0 is a zero of multiplicity 8, $\hat{\mu}_k^+$, $\hat{\mu}_k^-$, $\hat{\mu}_{-k}^+$ and $\hat{\mu}_{-k}^-$ are simple zeros of $\phi_0$, then it follows from (4.183), (4.184), (4.185), (4.187), Remark 4.27 and Rouché's theorem that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$ and the images of zeros of $\phi_0$ by the rotation of angle $\frac{\pi}{2}$ and these asymptotics are

$$\left.\begin{array}{l} \mu_k^{\pm} = \pm(4k-7)\frac{\pi}{4a} + o(1), \text{ where } k \geq k_0(\varepsilon) \text{ and} \\ \mu_k^{\pm} = \pm i(4|k|-7)\frac{\pi}{4a} + o(1), \text{ where } k \leq -k_0(\varepsilon) \end{array}\right\}, \tag{4.198}$$

with $o(1) \to 1$ as $|k| \to \infty$, see (4.182).

Let $S_k$, $k \in \mathbb{N}$ be the square defined in (4.66). Let

$$\phi_{11}(\mu) = -2i\alpha\mu^2 \cos(\mu a)\cosh(\mu a) \tag{4.199}$$

$$\phi_{12}(\mu) = -2i\alpha \sin(\mu a)\sinh(\mu a) \tag{4.200}$$

$$\phi_{13}(\mu) = -\frac{1}{\mu^3}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)). \tag{4.201}$$

It follows from (4.164) and (4.169) that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_0 = \max\left\{\hat{k}_0, \frac{a}{\pi}\sqrt[3]{\frac{13}{\alpha}}\right\}$

$$\begin{aligned} \frac{\alpha^2}{3}|\phi_0(\mu)| &= \frac{\alpha^2}{3}|\mu|^5|\cos(\mu a)\cosh(\mu a)||\tanh(\mu a) - \tan(\mu a)| \\ &= \frac{\alpha^2}{3}|\mu|^5|\cos(\mu a)\cosh(\mu a)||\phi_3(\mu)| \\ &\geq \frac{\alpha^2}{6}|\mu|^5|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{\alpha|\mu|^3}{12}2\alpha|\mu|^2|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{\alpha|\mu|^3}{12}|\phi_{11}(\mu)| > |\phi_{11}(\mu)|. \end{aligned} \tag{4.202}$$

Using (4.152) and (4.162) we have for all $\mu$ on the square $S_k$ where $k \in \mathbb{N}$, $k \geq \widetilde{m}_0 = \max\left\{\frac{a}{\pi}\sqrt[5]{\frac{13}{\alpha}}, \widetilde{k}_0\right\}$

$$\begin{aligned} \frac{\alpha^2}{3}|\phi_0(\mu)| &= \frac{\alpha^2}{3}|\mu|^5|\sin(\mu a)\sinh(\mu a)||\coth(\mu a) - \cot(\mu a)| \\ &= \frac{\alpha^2}{3}|\mu|^5|\sin(\mu a)\sinh(\mu a)||\phi_2(\mu)| \\ &\geq \frac{\alpha^2}{6}|\mu|^5|\sin(\mu a)\sinh(\mu a)| \\ &= \frac{\alpha|\mu|^5}{12}2\alpha|\sin(\mu a)\sinh(\mu a)| \\ &= \frac{\alpha|\mu|^5}{12}|\phi_{12}(\mu)| > |\phi_{12}(\mu)|. \end{aligned} \tag{4.203}$$

There exists $m_0 = \frac{a}{\pi}\sqrt[8]{\frac{4}{\alpha^2}}$ such that for all $\mu$ on the square $S_k$, $k \in \mathbb{N}$ and $k \geq m_0$, $\frac{1}{\alpha^2|\mu|^3} < \frac{|\mu|^5}{3}$. Thus for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq m_0$

$$|\phi_{13}(\mu)| < \frac{\alpha^2}{3}|\phi_0(\mu)|. \tag{4.204}$$

It follows from (4.202), (4.203) and (4.204) that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \breve{m}_0 = \max\{m_0, \widetilde{m}_0, \hat{m}_0\}$

$$|\phi_1(\mu)| < \alpha^2|\phi_0(\mu)|. \tag{4.205}$$

Since the square $S_k$ is a closed curve, then (4.205) and Rouché's theorem imply that $\phi$ and $\phi_0$ have the same number of zeros inside the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \breve{m}_0$.

**Remark 4.31.** Let $k_1 = \max\{k_0(\varepsilon), \breve{m}_0\}$. As 0 is a zero of multiplicity 8 of $\phi_0$, while $\hat{\mu}_k^+$, $\hat{\mu}_k^-$, $\hat{\mu}_{-k}^+$ and $\hat{\mu}_{-k}^-$ are its simple zeros, then the number of zeros of $\phi_0$ and therefore of $\phi$ inside the squares $S_k$, where $k \geq k_1$, is $4k$. Thus we have the following proposition.

**Proposition 4.32.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2)*, where $B_1(y) = y(0)$, $B_2(y) = y''(0)$, $B_3y = y'(a) - i\alpha\lambda y''(a)$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\left.\begin{array}{l} \hat{\mu}_k^{\pm} = \pm(4k-7)\frac{\pi}{4a} + o(1), \quad if\ k > 0 \\ \hat{\mu}_k^{\pm} = \pm i(4|k|-7)\frac{\pi}{4a} + o(1), \quad if\ k < 0 \end{array}\right\}.$$

*In particular, there is an even number of pure imaginary eigenvalues.*

**Remark 4.33.** We give only the enumeration of the zeros $\hat{\mu}_k^{\pm}$ of $\phi$, since the remainder of the proof of the proposition is identical to the proof derived for the remainder of Proposition 4.25.

*Proof.* The function $\phi$ has $4k_1$ number of zeros inside the square $S_{k_1}$, since there is no nonzero zeros of $\phi_0$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, then no zero of $\phi$ has an asymptotic of elements of the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Thus the zeros of $\phi$ inside the square $S_{k_1}$ are the following

$$\hat{\mu}_1^{\pm} = \pm\left(\frac{-3\pi}{4a}\right) + o(1),\ \hat{\mu}_2^{\pm} = \pm\left(\frac{\pi}{4a}\right) + o(1), \ldots, \hat{\mu}_{k_1+1}^{\pm} = \pm(4k_1 - 3)\frac{\pi}{4a} + o(1)$$
$$\hat{\mu}_{-1}^{\pm} = \pm i\left(\frac{-3\pi}{4a}\right) + o(1),\ \hat{\mu}_{-2}^{\pm} = \pm i\left(\frac{\pi}{4a}\right) + o(1), \ldots, \hat{\mu}_{-(k_1+1)}^{\pm} = \pm i(4k_1 - 3)\frac{\pi}{4a} + o(1).$$

Hence

$$\hat{\mu}_k^{\pm} = \pm(4k-7)\frac{\pi}{4a} + o(1), \text{ where } 1 \le k \le k_1,$$
$$\hat{\mu}_{-k}^{\pm} = \pm i(4|k|-7)\frac{\pi}{4a} + o(1), \text{ where } -k_1 \le k \le -1.$$

Using the same approach as in the proof of Proposition 4.23, we can show that the zeros of $\phi$ for $|k| \ge k_1$ are

$$\hat{\mu}_k^{\pm}, \ k = k_1, k_1+1, \dots$$
$$-k_1, -k_1-1, \dots$$

and satisfy

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(4k-7)\frac{\pi}{4a} + o(1), \text{ where } k \ge k_1, \\ \hat{\mu}_k^{\pm} = \pm i(4|k|-7)\frac{\pi}{4a} + o(1), \text{ where } k \le -k_1, \end{cases}$$

see (4.198). □

## 4.5 The boundary terms $B_1y$ and $B_2y$ are the following: $B_1y = y(0)$ and $B_2y = y'(0)$

Using the canonical fundamental system, then

$$\begin{cases} B_1y_1 = y_1(0) = 1, \\ B_1y_2 = y_2(0) = 0, \\ B_1y_3 = y_3(0) = 0, \\ B_1y_4 = y_4(0) = 0, \end{cases} \qquad \begin{cases} B_2y_1 = y_1'(0) = 0, \\ B_2y_2 = y_2'(0) = 1, \\ B_2y_3 = y_3'(0) = 0, \\ B_2y_4 = y_4'(0) = 0. \end{cases}$$

It follows that the characteristic matrix of this particular boundary problem is

$$M_c = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ B_3y_1 & B_3y_2 & B_3y_3 & B_3y_4 \\ B_4y_1 & B_4y_2 & B_4y_3 & B_4y_4 \end{pmatrix}. \tag{4.206}$$

The determinant of the characteristic matrix $M_c$ gives the characteristic function of the differential equation (3.2). The shape of the matrix $M_c$ leads to a reduced characteristic matrix of the boundary value problem.

The reduced characteristic matrix of the boundary value problem is

$$M = \begin{pmatrix} B_3 \\ B_4 \end{pmatrix} \begin{pmatrix} y_3 & y_4 \end{pmatrix} = \begin{pmatrix} B_3 y_3 & B_3 y_4 \\ B_4 y_3 & B_4 y_4 \end{pmatrix}. \tag{4.207}$$

It is easy to check that $\det M_c = \det M$.

### 4.5.1 Asymptotic of the eigenvalues for $B_3 y = y''(a) + i\alpha\lambda y'(a)$ and $B_4 y = y^{(3)}(a) - i\alpha\lambda y(a)$

It follows from (4.207) that

$$\begin{aligned}
\det M &= B_3 y_3 B_4 y_4 - B_4 y_3 B_3 y_4 \\
&= (y_3''(a) + i\alpha\mu^2 y_3'(a))(y_4^{(3)}(a) - i\alpha\mu^2 y_4(a)) - (y_3^{(3)}(a) - i\alpha\mu^2 y_3(a))(y_4''(a) + i\alpha\mu^2 y_4'(a)) \\
&= y_3''(a) y_4^{(3)}(a) + \alpha^2\mu^4 y_3'(a) y_4(a) - y_3^{(3)}(a) y_4''(a) - \alpha^2\mu^4 y_3(a) y_4'(a) + i\alpha\mu^2 (y_3'(a) y_4^{(3)}(a) \\
&\quad - y_3''(a) y_4(a) - y_3^{(3)}(a) y_4'(a) + y_3(a) y_4''(a)) \\
&= y_3''(a) y_4^{(3)}(a) - y_3^{(3)}(a) y_4''(a) + \alpha^2\mu^4 y_3'(a) y_4(a) - \alpha^2\mu^4 y_3(a) y_4'(a) \\
&\quad + i\alpha\mu^2 (y_3'(a) y_4^{(3)}(a) - y_3''(a) y_4(a) + y_3(a) y_4''(a) - y_4'(a) y_3^{(3)}(a)).
\end{aligned} \tag{4.208}$$

We can derive from (4.21) and (4.25) the following

$$\begin{cases}
y_3(x) = y_4'(x), \\
y_3'(x) = \frac{1}{2\mu}\sin(\mu x) + \frac{1}{2\mu}\sinh(\mu x) = y_4''(x), \\
y_3''(x) = \frac{1}{2}\cos(\mu x) + \frac{1}{2}\cosh(\mu x) = y_4^{(3)}(x), \\
y_3^{(3)}(x) = -\frac{\mu}{2}\sin(\mu x) + \frac{\mu}{2}\sinh(\mu x) = \mu^4 y_4(x).
\end{cases} \tag{4.209}$$

Thus

$$\begin{aligned}
\det M &= (y_4^{(3)}(a))^2 - \mu^4 y_4(a) y_4''(a) + \alpha^2\mu^4 y_4''(a) y_4(a) - \alpha^2\mu^4 (y_4'(a))^2 \\
&+ i\alpha\mu^2 (y_4''(a) y_4^{(3)}(a) - y_4^{(3)}(a) y_4(a) + y_4'(a) y_4''(a) - \mu^4 y_4'(a) y_4(a)) \\
&= (y_4^{(3)}(a))^2 - (1-\alpha^2)\mu^4 y_4''(a) y_4(a) - \alpha^2\mu^4 (y_4'(a))^2 \\
&+ i\alpha\mu^2 (y_4''(a) y_4^{(3)}(a) - y_4^{(3)}(a) y_4(a) + y_4'(a) y_4''(a) - \mu^4 y_4'(a) y_4(a)). \qquad (4.210)
\end{aligned}$$

However

$$(y_4^{(3)}(a))^2 = \frac{1}{4}\cos^2(\mu a) + \frac{1}{2}\cos(\mu a)\cosh(\mu a) + \frac{1}{4}\cosh^2(\mu a),$$

see (4.33), while

$$(y_4'(a))^2 = \frac{1}{4\mu^4}\cos^2(\mu a) - \frac{1}{2\mu^4}\cos(\mu a)\cosh + \frac{1}{4\mu^4}\cosh^2(\mu a)$$

and

$$y_4'(a) y_4''(a) = \frac{1}{4\mu^3}\left(-\sin(\mu a)\cos(\mu a) - \sinh(\mu a)\cos(\mu a) + \sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cosh(\mu a)\right),$$

see (4.31) and (4.29). But

$$\begin{aligned}
y_4(a) y_4''(a) &= \left(-\frac{1}{2\mu^3}\sin(\mu a) + \frac{1}{2\mu^3}\sinh(\mu a)\right)\left(\frac{1}{2\mu}\sin(\mu a) + \frac{1}{2\mu}\sinh(\mu a)\right) \\
&= -\frac{1}{4\mu^4}\sin^2(\mu a) + \frac{1}{4\mu^4}\sinh^2(\mu a), \qquad (4.211)
\end{aligned}$$

$$\begin{aligned}
y_4(a) y_4'(a) &= \left(-\frac{1}{2\mu^3}\sin(\mu a) + \frac{1}{2\mu^3}\sinh(\mu a)\right)\left(-\frac{1}{2\mu^2}\cos(\mu a) + \frac{1}{2\mu^2}\cosh(\mu a)\right) \\
&= \frac{1}{4\mu^5}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^5}\sin(\mu a)\cosh(\mu a) \\
&- \frac{1}{4\mu^5}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu^5}\sinh(\mu a)\cosh(\mu a), \qquad (4.212)
\end{aligned}$$

$$\begin{aligned}
y_4^{(3)}(a) y_4''(a) &= \left(\frac{1}{2}\cos(\mu a) + \frac{1}{2}\cosh(\mu a)\right)\left(\frac{1}{2\mu}\sin(\mu a) + \frac{1}{2\mu}\sinh(\mu a)\right) \\
&= \frac{1}{4\mu}\sin(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sinh(\mu a)\cos(\mu a) \\
&+ \frac{1}{4\mu}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu}\sinh(\mu a)\cosh(\mu a), \qquad (4.213)
\end{aligned}$$

$$
\begin{aligned}
y_4^{(3)}(a)y_4(a) &= \left(\frac{1}{2}\cos(\mu a) + \frac{1}{2}\cosh(\mu a)\right)\left(-\frac{1}{2\mu^3}\sin(\mu a) + \frac{1}{2\mu^3}\sinh(\mu a)\right) \\
&= -\frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) \\
&\quad - \frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a).
\end{aligned} \tag{4.214}
$$

Let

$$
F_0(a) = (y_4^{(3)}(a))^2 - (1-\alpha^2)\mu^4 y_4''(a)y_4(a) - \alpha^2\mu^4(y_4'(a))^2. \tag{4.215}
$$

Then

$$
\begin{aligned}
F_0(a) &= \frac{1}{4}\cos^2(\mu a) + \frac{1}{2}\cos(\mu a)\cosh(\mu a) + \frac{1}{4}\cosh^2(\mu a) \\
&\quad + (1-\alpha^2)\left(\frac{1}{4}\sin^2(\mu a) - \frac{1}{4}\sinh^2(\mu a)\right) \\
&\quad - \alpha^2\left(\frac{1}{4}\cos^2(\mu a) - \frac{1}{2}\cos(\mu a)\cosh(\mu a) + \frac{1}{4}\cosh^2(\mu a)\right) && (4.216) \\
&= \frac{1}{2}(1-\alpha^2) + \frac{1}{2}(1+\alpha^2)\cos(\mu a)\cosh(\mu a) && (4.217)
\end{aligned}
$$

Let

$$
F_1(a) = y_4''(a)y_4^{(3)}(a) - y_4^{(3)}(a)y_4(a) + y_4'(a)y_4''(a) - \mu^4 y_4'(a)y_4(a). \tag{4.218}
$$

Then it follows from (4.213), (4.214), (4.29), (4.212) and (4.218) that

$$
\begin{aligned}
F_1(a) &= \frac{1}{4\mu}\sin(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sin(\mu a)\cosh(\mu a) \\
&\quad + \frac{1}{4\mu}\sinh(\mu a)\cosh(\mu a) + \frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) \\
&\quad + \frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a) - \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a) - \frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) \\
&\quad - \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a) \\
&\quad - \frac{1}{4\mu}\sin(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sin(\mu a)\cosh(\mu a) \\
&\quad + \frac{1}{4\mu}\sinh(\mu a)\cos(\mu a) - \frac{1}{4\mu}\sinh(\mu a)\cosh(\mu a).
\end{aligned}
$$

Thus

$$
\begin{aligned}
F_1(a) &= \frac{1}{2\mu}(\sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cos(\mu a)) - \frac{1}{2\mu^3}(\sinh(\mu a)\cos(\mu a) \\
&\quad - \sin(\mu a)\cosh(\mu a)).
\end{aligned} \tag{4.219}
$$

It follows from (4.210), (4.216) and (4.219) that

$$\begin{aligned}\det M = {} & \frac{i\alpha}{2}\mu(\sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cos(\mu a)) \\ & + \frac{1}{2}[(1-\alpha^2) + (1+\alpha^2)\cos(\mu a)\cosh(\mu a)] \\ & - \frac{i\alpha}{2\mu}(\sinh(\mu a)\cos(\mu a) - \sin(\mu a)\cosh(\mu a)). \end{aligned} \tag{4.220}$$

Thus the characteristic equation $-2i \det M = 0$ is

$$\phi(\mu) = \alpha\phi_0(\mu) + \phi_1(\mu) = 0, \tag{4.221}$$

where

$$\phi_0(\mu) = \mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)), \tag{4.222}$$

$$\begin{aligned}\phi_1(\mu) = {} & -i[(1-\alpha^2) + (1+\alpha^2)\cos(\mu a)\cosh(\mu a)] \\ & - \frac{\alpha}{\mu}(\sinh(\mu a)\cos(\mu a) - \sin(\mu a)\cosh(\mu a)). \end{aligned} \tag{4.223}$$

The zeros of the function $\phi_0$ are 0 and the zeros of $\sin(\mu a)\cosh(\mu a)+\cos(\mu a)\sinh(\mu a)$. Observe that for $\mu \neq 0$, $\phi_0(\mu) = 0$ implies that $\cos(\mu a) \neq 0$ and $\cosh(\mu a) \neq 0$. Hence

$$\phi_0(\mu) = \mu\cos(\mu a)\cosh(\mu a)(\tan(\mu a) + \tanh(\mu a)),$$

therefore the nonzero zeros of $\phi_0$ are those $\mu \neq 0$ for which $\tan(\mu a) + \tanh(\mu a) = 0$. Since $\tan'(\mu a) \geq 1$ and $\tanh'(\mu a) > 0$ for $x \in \mathbb{R}$, the function $x \mapsto \tan(\mu a) + \tanh(\mu a)$ is increasing with positive derivative on each interval $\left(\left(k - \frac{1}{2}\right)\pi, \left(k + \frac{1}{2}\right)\pi\right)$. On each of these intervals, the function moves from $-\infty$ to $\infty$, thus we have exactly one simple zero $\hat{\mu}_k^{\pm}$ of $\tan(\mu a)+\tanh(\mu a)$ in each interval $\left(\left(\pm k - \frac{1}{2}\right)\pi, \left(\pm k + \frac{1}{2}\right)\pi\right)$, where $k$ is a positive integer, and no nonzero zero in $\left(-\frac{\pi}{2a}, \frac{\pi}{2a}\right)$. As the functions $x \mapsto \tan(\mu a)$ and $x \mapsto \tanh(\mu a)$ are odd functions, then the function $x \mapsto \tan(x) + \tanh(x)$, $x \in \mathbb{R}$ is an odd function. Whence $\hat{\mu}_k^- = -\hat{\mu}_k^+$, and since $\tanh(x) \to 1$ as $x \to \infty$, we have

$$\hat{\mu}_k^+ = (4k-1)\frac{\pi}{4a} + o(1),\ \hat{\mu}_k^- = -(4k-1)\frac{\pi}{4a} + o(1),\ k = 1, 2, \ldots. \tag{4.224}$$

As $\tan(i\gamma a) = i\tanh(\gamma a)$ and $\tanh(i\gamma a) = i\tan(\gamma a)$, the nonzero pure imaginary zeros of $\phi_0$ are simple and of the form

$$\hat{\mu}_{-k}^+ = i(4k-1)\frac{\pi}{4a} + o(1),\ \hat{\mu}_{-k}^- = -i(4k-1)\frac{\pi}{4a} + o(1),\ k = 1, 2, \ldots. \tag{4.225}$$

The function $\phi_0$ in this subsection is obtained by multiplying the function $\phi_0$ in Subsection 4.4.3 by $\mu^{-2}$ and by replacing the trigonometric functions $\sin(\mu a)$ and $\cos(\mu a)$ respectively with $\sin(-\mu a)$ and $\cos(-\mu a)$. This leads to the equation (4.128) for $\mu \neq 0$. Hence Remark 4.26 implies that the zeros of $\phi_0$ are either real or pure imaginary.

Thus the zeros of $\phi_0$ are the following

$$0,\ \hat{\mu}_k^{\pm} = \pm(4k-1)\frac{\pi}{4a} + o(1),\ \hat{\mu}_{-k}^{\pm} = \pm i(4k-1)\frac{\pi}{4a} + o(1),\ k = 1, 2, \dots. \tag{4.226}$$

Let $\widetilde{\psi}_0(\mu) = \mu$ and

$$\widetilde{\psi}_1(\mu) = \tan(\mu a) + \tanh(\mu a). \tag{4.227}$$

Then $\widetilde{\psi}_1'(\mu) = a(1 + \tan^2(\mu a)) + \frac{a}{\cosh^2(\mu a)} > 0$. Thus it follows that 0 is a simple zero of $\psi_0$ and $\widetilde{\psi}_1$ therefore 0 is a zero of multiplicity 2 of $\phi_0$. Whence the zeros of $\phi_0$ counted with multiplicity are

$$\left.\begin{array}{l} \hat{\mu}_0^{\pm} = 0,\ \hat{\mu}_k^{\pm} = \pm(4k-1)\frac{\pi}{4a} + o(1), \\ \hat{\mu}_{-k}^{\pm} = \pm i(4k-1)\frac{\pi}{4a} + o(1),\ k = 1, 2, \dots \end{array}\right\}, \tag{4.228}$$

see (4.226).

Let

$$\phi_{00}(\mu) = \cos(\mu a) + \sin(\mu a) \tag{4.229}$$

$$\begin{aligned}\phi_{01}(\mu) &= (-1 + \tanh(\mu a))\cos(\mu a) - \frac{i}{\alpha\mu}\left(\frac{1-\alpha^2}{\cosh(\mu a)} + (1+\alpha^2)\cos(\mu a)\right) \\ &\quad - \frac{1}{\mu^2}(-\sin(\mu a) + \cos(\mu a)\tanh(\mu a)).\end{aligned} \tag{4.230}$$

Then

$$\phi_{02}(\mu) = \frac{\phi(\mu)}{\alpha\mu\cosh(\mu a)} = \phi_{00}(\mu) + \phi_{01}(\mu). \tag{4.231}$$

It is easy to see that

$$\mu_k^{00} = \left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right),\ k \in \mathbb{Z} \text{ are the zeros of } \phi_{00} \tag{4.232}$$

and

$$\widetilde{\mu}_k^{00} = i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right),\ k \in \mathbb{Z} \text{ are the images of } \mu_k^{00} \text{ by the rotation of angle } \frac{\pi}{2}. \tag{4.233}$$

Let

$$\left.\begin{array}{l} R_k \text{ be the rectangles with vertices } (4k-1)\frac{\pi}{4a} \pm \varepsilon \pm i\varepsilon,\ k \in \mathbb{Z} \text{ and } \varepsilon \in (0, \frac{\pi}{2a}) \\ R_{-k} \text{ their symmetric images with respect to the } y \text{ axis },\ \widetilde{R}_k,\ \text{and } \widetilde{R}_{-k}, \\ \text{the respective images of the rectangles } R_k \text{ and } R_{-k} \text{ by the rotation} \\ \text{of angle } \frac{\pi}{2} \end{array}\right\}. \quad (4.234)$$

**Remark 4.34.** We can observe that $\mu_k^{00+} = \left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in R_k$, $-\mu_k^{00+} = -\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in R_{-k}$, $\widetilde{\mu}_k^{00+} = i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_k$ and $-\widetilde{\mu}_k^{00+} = -i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_{-k}$.

Since $\varepsilon < \frac{\pi}{2a}$, the rectangles $R_k$, $k \in \mathbb{Z}$ do not intersect, as well as the rectangles $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$. As $|\phi_{00}|$ is periodic of period $\frac{\pi}{a}$, there exists a constant $\rho(\varepsilon) > 0$ such that $|\phi_{00}(\mu)| > \rho(\varepsilon)$ for all $\mu$ on the rectangle $R_k$, $k \in \mathbb{Z}$. On the other hand for all $\mu$ on the rectangle $R_k$, where $|k| \geq k_1(\varepsilon)$ is sufficiently large positive, we have

$$\left.\begin{array}{l} |\cos(\mu a)(-1 + \tanh(\mu a)| = \left|\frac{e^{(y-ix)} + e^{-(y-ix)}}{e^{2\mu a} + 1}\right| \leq \frac{2e^{\Im|\mu a|}}{e^{2|\Re \mu a|}} < 3e^{-|\Re \mu a|} \\ |\cos(\mu a)| < \sqrt{2}, \quad |\sin(\mu a)| < \sqrt{2}, \quad |\tanh(\mu a)| < \sqrt{2} \\ |\sin(\mu a) - \cos(\mu a)\tanh(\mu a)| < 3\sqrt{2} \end{array}\right\}, \quad (4.235)$$

see (4.138). We, also, have for sufficiently large positive $|k| \geq k_2(\varepsilon)$ and for all $\mu$ on the rectangle $R_k$

$$\left|\frac{1}{\cosh(\mu a)}\right| = \frac{2}{|e^{\mu a} + e^{-\mu a}|} < \frac{2}{e^{|\Re \mu a|}} < 3e^{-|\Re \mu a|}. \quad (4.236)$$

Thus

$$\left|\frac{1-\alpha^2}{\cosh(\mu a)}\right| = \frac{2|1-\alpha^2|}{|e^{\mu a} + e^{-\mu a}|} < \frac{2(1+\alpha^2)}{e^{|\Re \mu a|}} < 3(1+\alpha^2)e^{-|\Re \mu a|}.$$

Let $k_0(\varepsilon) = \max\{k_1(\varepsilon), k_2(\varepsilon)\}$. Then it follows from (4.230), (4.235) and (4.236) that

$$|\phi_{01}(\mu)| < \frac{(1+\alpha^2)(3+\sqrt{2})}{\alpha|\mu|} + \frac{3\sqrt{2}}{|\mu|^2} + 3e^{-|\Re \mu a|} \quad (4.237)$$

for all $\mu$ on the rectangle $R_k$ and $|k| \geq k_0(\varepsilon)$ large enough. Since the right hand tends to 0 as $|\Re \mu a| \to \infty$, then for all $\mu$ on the rectangle $R_k$, where $k \in \mathbb{Z}$, $|k| > k_0(\varepsilon)$,

$$|\phi_{01}(\mu)| < |\phi_{00}(\mu)|. \quad (4.238)$$

For all $\mu \in \mathbb{C}$, we have

$$\begin{aligned}\phi_0(-\mu) &= -\mu(\sin(-\mu a)\cosh(-\mu a) + \cos(-\mu a)\sinh(-\mu a)) \\ &= -\mu(-\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \\ &= \mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) = \phi_0(\mu), \end{aligned} \tag{4.239}$$

while

$$\begin{aligned}\phi_1(-\mu) &= -i[(1-\alpha^2) + (1+\alpha^2)\cos(-\mu a)\cosh(-\mu a)] \\ &+ \frac{\alpha}{\mu}(\sinh(-\mu a)\cos(-\mu a) - \sin(-\mu a)\cosh(-\mu a)) \\ &= -i[(1-\alpha^2) + (1+\alpha^2)\cos(\mu a)\cosh(\mu a)] \\ &+ \frac{\alpha}{\mu}(-\sinh(\mu a)\cos(\mu a) + \sin(\mu a)\cosh(\mu a)) \\ &= -i[(1-\alpha^2) + (1+\alpha^2)\cos(\mu a)\cosh(\mu a)] \\ &- \frac{\alpha}{\mu}(\sinh(\mu a)\cos(\mu a) - \sin(\mu a)\cosh(\mu a)) = \phi_1(\mu). \end{aligned} \tag{4.240}$$

Thus it follows from (4.221), (4.239) and (4.240) that $\phi$ is an even function and therefore we have the same estimates (4.237) and (4.238) for all $\mu$ on the rectangle $R_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$.

Let

$$\begin{aligned}\widetilde{\phi}_0(\mu) &= \alpha\mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \\ &- \frac{\alpha}{\mu}(\sinh(\mu a)\cos(\mu a) - \sin(\mu a)\cosh(\mu a)) \end{aligned} \tag{4.241}$$

$$\widetilde{\phi}_1(\mu) = -i[(1-\alpha^2) + (1+\alpha^2)\cos(\mu a)\cosh(\mu a)]. \tag{4.242}$$

Then

$$\phi(\mu) = \widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu) \tag{4.243}$$

$$\begin{aligned}\widetilde{\phi}_0(i\mu) &= i\alpha\mu(\sin(i\mu a)\cosh(i\mu a) + \cos(i\mu a)\sinh(i\mu a)) \\ &\quad - \frac{\alpha}{i\mu}(\sinh(i\mu a)\cos(i\mu a) - \sin(i\mu a)\cosh(i\mu a)) \\ &= i\alpha\mu(i\sinh(\mu a)\cos(\mu a) + i\cosh(\mu a)\sin(\mu a)) \\ &\quad - \frac{\alpha}{i\mu}(i\sin(\mu a)\cosh(\mu a) - i\sinh(\mu a)\cos(\mu a)) \\ &= -\alpha\mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \\ &\quad + \frac{\alpha}{\mu}(\sinh(\mu a)\cos(\mu a) - \sin(\mu a)\cosh(\mu a)) = -\widetilde{\phi}_0(\mu), \end{aligned} \tag{4.244}$$

while

$$\begin{aligned}\widetilde{\phi}_1(i\mu) &= -i[(1-\alpha^2) + (1+\alpha^2)\cos(i\mu a)\cosh(i\mu a)] \\ &= -i[(1+\alpha^2) + (1-\alpha^2)\cos(\mu a)\cosh(\mu a)] = \widetilde{\phi}_1(\mu). \end{aligned} \tag{4.245}$$

Thus

$$\phi(i\mu) = -\widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu). \tag{4.246}$$

Using the equations (4.221), (4.222) and (4.223), we can obtain the same estimates (4.237) and (4.238) for all $\mu$ on the rectangles $R_k$ and $R_{-k}$ where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough. Since $|\phi(\mu)|$ and $|\phi(i\mu)|$ have the same upper bound $|\widetilde{\phi}_0(\mu)|+|\widetilde{\phi}_1(\mu)|$ for all $\mu$ on the rectangles $R_k$ and $R_{-k}$, see (4.243) and (4.246), then we can also obtain the same estimates (4.237) and (4.238) for all $\mu$ on the rectangles $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ by using the same equations (4.221), (4.222) and (4.223).

Since the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ are closed curves, 0 is a zero of multiplicity 2 of $\phi_0$, $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$ are simple zeros of $\phi_0$ and $\phi_0$ has no nonzero zero in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, then it follows from (4.238), Remark 4.34 and Rouché's theorem that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$, where the asymptotics of the zeros of $\phi_0$ are

$$\left.\begin{aligned} \hat{\mu}_k^{\pm} &= \pm(4k-1)\tfrac{\pi}{4a} + o(1), \text{ where } k \in \mathbb{Z},\ k \geq k_0(\varepsilon) \text{ and} \\ \hat{\mu}_k^{\pm} &= \pm i(4|k|-1)\tfrac{\pi}{4a} + o(1), \text{ where } k \in \mathbb{Z},\ k \leq -k_0(\varepsilon) \end{aligned}\right\}, \tag{4.247}$$

with $o(1) \to 0$ as $|k| \to \infty$, see (4.228).

Let $S_k$, $k \in \mathbb{N}$ be the square defined in (4.66) and $\widetilde{\psi}_1$ be the function defined in (4.227). Then for $k \in \mathbb{Z}$ and $\mu = \frac{k\pi}{a} + i\gamma$, where $\gamma \in \mathbb{R}$,

$$|\tan(\mu a) \pm 1| \geq 1,$$

see (4.165), also there exists $\widetilde{k}_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq \widetilde{k}_0$ and $\gamma \in \mathbb{R}$

$$\left|\tanh\left(\left(k\frac{\pi}{a} + i\gamma\right)a\right) - \operatorname{sgn}(k)\right| < \frac{1}{2},$$

see (4.166). Thus

$$|\widetilde{\psi}_1(\mu)| \geq \frac{1}{2} \text{ for } \mu = \frac{k\pi}{a} + i\gamma,\ k \in \mathbb{Z},\ |k| \geq \widetilde{k}_0 \text{ and } \gamma \in \mathbb{R}, \tag{4.248}$$

see (4.167). By interchanging tan and tanh for $\mu = \gamma + ik\frac{\pi}{a}$ we obtain the same estimate (4.248). Therefore

$$|\widetilde{\psi}_1(\mu)| \geq \frac{1}{2} \text{ for all } \mu \text{ on the square with vertices } \pm k\frac{\pi}{a} \pm ik\frac{\pi}{a}, \text{ where } k \in \mathbb{N}, k \geq \widetilde{k}_0. \tag{4.249}$$

Let

$$\phi_{11}(\mu) = -\frac{\alpha}{\mu}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) \tag{4.250}$$

$$\phi_{12}(\mu) = -(1+\alpha^2)i\cos(\mu a)\cosh(\mu a) \tag{4.251}$$

$$\phi_{13}(\mu) = -(1-\alpha^2)i. \tag{4.252}$$

The function $x \to \tan(x) - \tanh(x)$ is continuous for all $x \in \mathbb{R}$, where $x \neq \frac{\pi}{2} + k\pi$ and $k \in \mathbb{Z}$. Thus there exist $M_0 > 0$, such that

$$|\tan(\mu a) - \tanh(\mu a)| \leq M_0 \text{ for all } \mu \text{ on the square } S_k, \text{where } k \in \mathbb{N}. \tag{4.253}$$

Then for $\mu$ on the square with vertices $\pm k\frac{\pi}{a} \pm ik\frac{\pi}{a}$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_0 = \max\left\{\widetilde{k}_0, \frac{a}{\pi}\sqrt{7M_0}\right\}$

$$\begin{aligned}
|\phi_{11}(\mu)| &= \frac{\alpha}{|\mu|}|\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)| \\
&= \frac{\alpha}{|\mu|}|\cos(\mu a)\cosh(\mu a)||\tan(\mu a) - \tanh(\mu a)| \\
&\leq \frac{2\alpha M_0}{2|\mu|}|\cos(\mu a)\cosh(\mu a)| \\
&\leq \frac{2\alpha M_0}{|\mu|}|\cos(\mu a)\cosh(\mu a)||\tan(\mu a) + \tanh(\mu a)| \\
&\leq \frac{6M_0}{|\mu|^2}\frac{\alpha}{3}|\mu||\cos(\mu a)\cosh(\mu a)||\tan(\mu a) + \tanh(\mu a)| \\
&\leq \frac{6M_0}{|\mu|^2}\frac{\alpha}{3}|\phi_0(\mu)| < \frac{\alpha}{3}|\phi_0(\mu)|.
\end{aligned} \tag{4.254}$$

On the other hand, for $\mu$ on the square with vertices $\pm k\frac{\pi}{a} \pm ik\frac{\pi}{a}$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_0 = \max\left\{\widetilde{k}_0, \frac{7a(1+\alpha^2)}{\alpha\pi}\right\}$, we have

$$\begin{aligned}
\frac{\alpha}{3}|\phi_0(\mu)| &= \frac{\alpha}{3}|\mu||\widetilde{\psi}_1(\mu)||\cos(\mu a)\cosh(\mu a)| \\
&\geq \frac{\alpha}{6}|\mu|\frac{|1+\alpha^2|}{|1+\alpha^2|}|\cos(\mu a)\cosh(\mu a)| \\
&\geq \frac{\alpha|\mu|}{6|1+\alpha^2|}|1+\alpha^2||\cos(\mu a)\cosh(\mu a)| \\
&\geq \frac{\alpha|\mu|}{|6(1+\alpha^2)|}|\phi_{12}(\mu)| \\
&> |\phi_{12}(\mu)|.
\end{aligned} \tag{4.255}$$

Putting $\mu a = x + iy$, we have

$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y, \tag{4.256}$$

$$\cosh(x+iy) = \cos y \cosh x + i \sin y \sinh x. \tag{4.257}$$

Thus

$$\begin{aligned}
|\cos(x+iy)\cosh(x+iy)| &= |\cos(x+iy)||\cosh(x+iy)| \\
&= \left(\sqrt{\cos^2 x + \sinh^2 y}\right)\left(\sqrt{\cos^2 y + \sinh^2 x}\right) \geq |\sinh x|.
\end{aligned} \tag{4.258}$$

Hence there exists $\widehat{k}_0 = \frac{a\ln 8}{\pi}$, such that for $\mu$ on the square with vertices $\pm k\frac{\pi}{a} \pm ik\frac{\pi}{a}$, $|\cos(\mu a)\cosh(\mu a)| > 3$. Thus for $\mu$ on the square with vertices $\pm k\frac{\pi}{a} \pm ik\frac{\pi}{a}$, where $k \in \mathbb{N}$ and $k \geq \widehat{m}_0 = \max\left\{\widetilde{k}_0, \widehat{k}_0, \frac{2a(1+\alpha^2)}{\alpha\pi}\right\}$, we have

$$\begin{aligned}
|\phi_{13}(\mu)| &< 1 + \alpha^2 \\
&< \frac{\alpha}{3}\frac{2(1+\alpha^2)}{\alpha|\mu|}|\mu||\cos(\mu a)\cosh(\mu a)||\tan(\mu a) + \tanh(\mu a)| < \frac{\alpha}{3}|\phi_0(\mu)|.
\end{aligned} \tag{4.259}$$

Let $\breve{m}_0 = \max\{\hat{m}_0, \widetilde{m}_0, \widehat{m}_0\}$. Then it follows from (4.254), (4.255) and (4.259), that for $\mu$ on the square with vertices $\pm k\frac{\pi}{a} \pm ik\frac{\pi}{a}$, where $k \in \mathbb{N}$ and $k \geq \breve{m}_0$,

$$|\phi_1(\mu)| < \alpha|\phi_0(\mu)|. \tag{4.260}$$

As the square $S_k$ is a closed curve, then (4.260) and Rouch'e's theorem imply that $\phi$ and $\phi_0$ have the same number of zeros inside the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \breve{m}_0$.

**Remark 4.35.** Let $k_1 = \max\{\breve{m}_0, k_0(\varepsilon)\}$. Since 0 is a zero of multiplicity 2 of $\phi_0$, $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$ are simple zeros, then the number of zeros of $\phi_0$ and therefore of $\phi$ inside the square $S_k$, where $k \in \mathbb{N}$ and $k \geq k_1$, is $4k + 2$.

**Proposition 4.36.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2)*, where $B_1(y) = y(0)$, $B_2(y) = y'(0)$, $B_3 y = y''(a) + i\alpha\lambda y'(a)$ and $B_4 y = y^{(3)}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{cases} \hat{\mu}_k^{\pm} & = \pm(4k-1)\frac{\pi}{4a} + o(1), \quad if\ k > 0, \\ \hat{\mu}_k^{\pm} & = \pm i(4|k|-1)\frac{\pi}{4a} + o(1), \quad if\ k < 0. \end{cases}$$

*In particular, there is an odd number of pure imaginary eigenvalues.*

**Remark 4.37.** We give only the enumeration of the zeros $\hat{\mu}_k^{\pm}$ of $\phi$. The remainder of the proof is identical to the remainder of the proof derived for Proposition 4.23.

*Proof.* The function $\phi$ has $4k_1 + 2$ number of zeros inside the square $S_{k_1}$, since there is no nonzero zero of $\phi_0$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, then there is no nonzero of $\phi$ inside this interval. Thus the zeros of $\phi$ inside the square $S_{k_1}$ are the following

$$\begin{aligned} &\hat{\mu}_0^{\pm} = \pm\frac{\pi}{4a} + o(1),\ \hat{\mu}_1^{\pm} = \pm\frac{3\pi}{4a} + o(1), \dots,\ \hat{\mu}_{k_1}^{\pm} = \pm(4k_1 - 1)\frac{\pi}{4a} + o(1), \\ &\hat{\mu}_{-1}^{\pm} = \pm i\frac{3\pi}{4a} + o(1),\ \hat{\mu}_{-2}^{\pm} = \pm i\frac{7\pi}{4a} + o(1), \dots,\ \hat{\mu}_{-k_1}^{\pm} = \pm i(4k_1 - 1)\frac{\pi}{4a} + o(1). \end{aligned}$$

Whence

$$\begin{aligned} \hat{\mu}_k^{\pm} &= \pm(4k-1)\frac{\pi}{4a} + o(1), \text{ where } 1 \leq k \leq k_1, \\ \hat{\mu}_k^{\pm} &= \pm i(4|k|-1)\frac{\pi}{4a} + o(1), \text{ where } -k_1 \leq k \leq -1. \end{aligned}$$

Using the approach of the proof of Proposition 4.23, we can prove that the zeros of $\phi$ for $|k| \geq k_0$ are

$$\begin{aligned} \hat{\mu}_k^{\pm},\ k &= k_0, k_0 + 1, \dots \\ &- k_0, -k_0 - 1, \dots \end{aligned}$$

where $k_0 = k_1 - n + 1$ and satisfy

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(4k-1)\frac{\pi}{4a} + o(1), \text{ where } k \geq k_0, \\ \hat{\mu}_k^{\pm} = \pm i(4|k|-1)\frac{\pi}{4a} + o(1), \text{ where } k \leq -k_0, \end{cases}$$

see (4.247). □

### 4.5.2 Asymptotic of the eigenvalues for $B_3 y = y''(a) + i\alpha\lambda y'(a)$ and $B_4 y = y(a) + i\alpha\lambda y^{(3)}(a)$

It follows from (4.207) that

$$\begin{aligned} \det M &= B_3 y_3 B_4 y_4 - B_4 y_3 B_3 y_4 \\ &= (y_3''(a) + i\alpha\mu^2 y_3'(a))(y_4(a) + i\alpha\mu^2 y_4^{(3)}(a)) - (y_3(a) + i\alpha\mu^2 y_3^{(3)}(a))(y_4''(a) + i\alpha\mu^2 y_4'(a)) \\ &= y_3''(a)y_4(a) - \alpha^2\mu^4 y_3'(a)y_4^{(3)}(a) - y_3(a)y_4''(a) + \alpha^2\mu^4 y_3^{(3)}(a)y_4'(a) + i\alpha\mu^2(y_3'(a)y_4(a) \\ &+ y_3''(a)y_4^{(3)}(a) - y_3(a)y_4'(a) - y_3^{(3)}(a)y_4''(a)) \\ &= y_3''(a)y_4(a) - y_3(a)y_4''(a) - \alpha^2\mu^4 y_3'(a)y_4^{(3)}(a) + \alpha^2\mu^4 y_3^{(3)}(a)y_4'(a) \\ &+ i\alpha\mu^2(y_3'(a)y_4(a) + y_3''(a)y_4^{(3)}(a) - y_3^{(3)}(a)y_4''(a) - y_4'(a)y_3(a)). \end{aligned} \tag{4.261}$$

Let

$$A_0(a) = y_3''(a)y_4(a) - y_3(a)y_4''(a) - \alpha^2\mu^4 y_3'(a)y_4^{(3)}(a) + \alpha^2\mu^4 y_3^{(3)}(a)y_4'(a) \tag{4.262}$$

and

$$A_1(a) = y_3'(a)y_4(a) + y_3''(a)y_4^{(3)}(a) - y_3^{(3)}(a)y_4''(a) - y_4'(a)y_3(a). \tag{4.263}$$

Then it follows from (4.209) that

$$A_0(a) = y_4^{(3)}(a)y_4(a) - y_4'(a)y_4''(a) - \alpha^2\mu^4 y_4''(a)y_4^{(3)}(a) + \alpha^2\mu^8 y_4(a)y_4'(a),$$

while

$$A_1(a) = y_4''(a)y_4(a) + (y_4^{(3)}(a))^2 - (y_4'(a))^2 - \mu^4 y_4(a)y_4''(a).$$

Thus (4.29), (4.212), (4.213) and (4.214) give

$$\begin{aligned}
A_0(a) &= -\frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) - \frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a) \\
&+ \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a) - \Big(-\frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) \\
&+ \frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a)) - \alpha^2\mu^4(\frac{1}{4\mu}\sin(\mu a)\cos(\mu a) \\
&+ \frac{1}{4\mu}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu}\sinh(\mu a)\cosh(\mu a)\Big) \\
&+ \alpha^2\mu^8\left(\frac{1}{4\mu^5}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^5}\sin(\mu a)\cosh(\mu a) - \frac{1}{4\mu^5}\sinh(\mu a)\cos(\mu a)\right. \\
&\left.+ \frac{1}{4\mu^5}\sinh(\mu a)\cosh(\mu a)\right) \\
&= \frac{1}{2\mu^3}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) - \frac{\alpha^2\mu^3}{2}(\cos(\mu a)\sinh(\mu a) \\
&+ \sin(\mu a)\cosh(\mu a)),
\end{aligned} \tag{4.264}$$

while from (4.31), (4.33) and (4.211) we have

$$\begin{aligned}
A_1(a) &= -\frac{1}{4\mu^4}\sin^2(\mu a) + \frac{1}{4\mu^4}\sinh^2(\mu a) + \frac{1}{4}\cos^2(\mu a) + \frac{1}{2}\cos(\mu a)\cosh(\mu a) \\
&+ \frac{1}{4}\cosh^2(\mu a) - \left(\frac{1}{4\mu^4}\cos^2(\mu a) - \frac{1}{2\mu^4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4\mu^4}\cosh^2(\mu a)\right) \\
&- \mu^4\left(-\frac{1}{4\mu^4}\sin^2(\mu a) + \frac{1}{4\mu^4}\sinh^2(\mu a)\right) \\
&= -\frac{1}{4\mu^4}(\cos^2(\mu a) + \sin^2(\mu a) + \cosh^2(\mu a) - \sinh^2(\mu a)) + \frac{1}{2\mu^4}\cos(\mu a)\cosh(\mu a) \\
&+ \frac{1}{4}(\cos^2(\mu a) + \sin^2(\mu a) + \cosh^2(\mu a) - \sinh^2(\mu a)) + \frac{1}{2}\cos(\mu a)\cosh(\mu a) \\
&= -\frac{1}{2\mu^4} + \frac{1}{2\mu^4}\cos(\mu a)\cosh(\mu a) + \frac{1}{2} + \frac{1}{2}\cos(\mu a)\cosh(\mu a).
\end{aligned} \tag{4.265}$$

Whence (4.261), (4.264) and (4.265) yield

$$\begin{aligned}
\det M &= -\frac{\alpha^2\mu^3}{2}(\cos(\mu a)\sinh(\mu a) + \sin(\mu a)\cosh(\mu a)) + \frac{i\alpha\mu^2}{2} + \frac{i\alpha\mu^2}{2}\cos(\mu a)\cosh(\mu a) \\
&+ \frac{1}{2\mu^3}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) - \frac{i\alpha}{2\mu^2} + \frac{i\alpha}{2\mu^2}\cos(\mu a)\cosh(\mu a).
\end{aligned} \tag{4.266}$$

It follows that the characteristic equation $-2\det M = 0$ is

$$\phi(\mu) = \alpha^2\phi_0(\mu) + \phi_1(\mu) = 0, \tag{4.267}$$

where

$$\phi_0(\mu) = \mu^3(\cos(\mu a)\sinh(\mu a) + \sin(\mu a)\cosh(\mu a)), \tag{4.268}$$

$$\begin{aligned}\phi_1(\mu) &= -i\alpha\mu^2 - i\alpha\mu^2\cos(\mu a)\cosh(\mu a) - \frac{1}{\mu^3}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a))\\ &\quad + \frac{i\alpha}{\mu^2}(1-\cos(\mu a)\cosh(\mu a)).\end{aligned} \tag{4.269}$$

The function $\phi_0$ in this subsection is obtained from $\phi_0$ in Subsection 4.4.3 by replacing the trigonometric functions $\sin(\mu a)$ and $\cos(\mu a)$ respectively with $\sin(-\mu a)$ and $\cos(-\mu a)$. This leads to the equation (4.128) for $\mu \neq 0$. Therefore it follows from Remark 4.26 that the zeros of $\phi_0$ are either real or pure imaginary.

The zeros of the function $\phi_0$ are identical to the zeros of $\phi_0$ defined in Subsection 4.5.1. These zeros are 0, $\hat{\mu}_k^{\pm} = \pm(4k-5)\frac{\pi}{4a} + o(1)$ and $\hat{\mu}_{-k}^{\pm} = \pm i(4k-5)\frac{\pi}{4a} + o(1)$, see (4.268), (4.224) and (4.225). We recall that 0 is a zero of multiplicity 3 of $\mu \mapsto \psi_0(\mu) = \mu^3$, while it is a simple zero of $\mu \mapsto \psi_1(\mu) = \cos(\mu a)\sinh(\mu a) + \sin(\mu a)\cosh(\mu a)$. Thus 0 is a zero of multiplicity 4 of $\phi_0$, while $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$ are its simple zeros. Whence the zeros of $\phi_0$ counted with multiplicity are the following

$$\left.\begin{aligned}&\hat{\mu}_{-1}^{\pm} = 0,\ \hat{\mu}_1^{\pm} = 0,\ \hat{\mu}_k^{\pm} = \pm(4k-5)\tfrac{\pi}{4a} + o(1),\\ &\hat{\mu}_{-k}^{\pm} = \pm i(4k-5)\tfrac{\pi}{4a} + o(1),\ k = 2,3,\dots\end{aligned}\right\}. \tag{4.270}$$

Let

$$\phi_{00}(\mu) = \cos(\mu a) + \sin(\mu a) \tag{4.271}$$

$$\begin{aligned}\phi_{01}(\mu) &= (-1+\tanh(\mu a))\cos(\mu a) - \frac{i}{\alpha\mu\cosh(\mu a)} - \frac{i\cos(\mu a)}{\alpha\mu}\\ &\quad - \frac{1}{\alpha^2\mu^6}(\cos(\mu a)\tanh(\mu a) - \sin(\mu a)) + \frac{i}{\alpha\mu^5}\left(\frac{1}{\cosh(\mu a)} - \cos(\mu a)\right).\end{aligned} \tag{4.272}$$

Then

$$\phi_{02}(\mu a) := \frac{\phi(\mu)}{\alpha^2\mu^3\cosh(\mu a)} = \phi_{00}(\mu) + \phi_{01}(\mu). \tag{4.273}$$

We recall that

$$\mu_k^{00} = \left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right),\ k \in \mathbb{Z}$$

are the zeros of $\phi_{00}$, see (4.229), (4.232), (4.271), while

$$\widetilde{\mu}_k^{00} = i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right),\ k \in \mathbb{Z}$$

are the images of these zeros by the rotation of angle $\frac{\pi}{2}$.

Let $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{Z}$ be the rectangles defined in (4.234).

As $\varepsilon < \frac{\pi}{2a}$, the rectangles $R_k$, $k \in \mathbb{Z}$ do not intersect, as well as the rectangles $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{Z}$. Since $|\phi_{00}|$ is periodic of period $\frac{\pi}{a}$, there exist a constant $\rho(\varepsilon) > 0$ such that $|\phi_{00}(\mu)| > \rho(\varepsilon)$ for all $\mu$ on the rectangle $R_k$, $k \in \mathbb{Z}$. For all $\mu$ on the rectangle $R_k$, where $|k| \geq k_1(\varepsilon)$ sufficiently large positive we have,

$$\left.\begin{array}{l} |\cos(\mu a)(-1+\tanh(\mu a)| = \left|\frac{e^{(y-ix)}+e^{-(y-ix)}}{e^{2\mu a}+1}\right| \leq \frac{2e^{\Im|\mu a|}}{e^{2|\Re\mu a|}} < 3e^{-|\Re\mu a|} \\ |\cos(\mu a)| < \sqrt{2}, \quad |\sin(\mu a)| < \sqrt{2} \quad |\tanh(\mu a)| < \sqrt{2} \\ |\sin(\mu a) - \cos(\mu a)\tanh(\mu a)| < 3\sqrt{2} \end{array}\right\},$$

see (4.138) and

$$\left|\frac{1}{\cosh(\mu a)}\right| < 3e^{-|\Re\mu a|},$$

for all $\mu$ on the rectangle $R_k$, where $|k| \geq k_2(\varepsilon)$ is large enough, see (4.236). Let $k_0(\varepsilon) = \max\{k_1(\varepsilon), k_2(\varepsilon)\}$. Then it follows from (4.138) and (4.236) that

$$|\phi_{01}(\mu)| < \frac{3+\sqrt{2}}{\alpha|\mu|} + \frac{3+\sqrt{2}}{\alpha|\mu|^5} + \frac{3\sqrt{2}}{\alpha^2|\mu|^6} + 3e^{-|\Re\mu a|} \tag{4.274}$$

for $\mu$ on the rectangle $R_k$ where $k \in \mathbb{N}$ and $k \geq k_0(\varepsilon)$ large enough. Since the right hand side tends to 0 as $|\Re\mu a| \to \infty$, it follows that for all $\mu$ on the rectangle $R_k$, where $k \in \mathbb{Z}$ and $|k| > k_0(\varepsilon)$ large enough,

$$|\phi_{01}(\mu)| < |\phi_{00}(\mu)|. \tag{4.275}$$

For all $\mu \in \mathbb{C}$, we have

$$\begin{aligned} \phi_0(-\mu) &= (-\mu)^3(\cos(-\mu a)\sinh(-\mu a) + \sin(-\mu a)\cosh(-\mu a)) \\ &= -\mu^3(-\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) \\ &= \mu^3(\cos(\mu a)\sinh(\mu a) + \sin(\mu a)\cosh(\mu a)) = \phi_0(\mu), \end{aligned} \tag{4.276}$$

$$\begin{aligned}
\phi_1(-\mu) &= -i\alpha(-\mu)^2 - i\alpha(-\mu)^2\cos(-\mu a)\cosh(-\mu a) - \frac{1}{(-\mu)^3}(\cos(-\mu a)\sinh(-\mu a) \\
&\quad - \sin(-\mu a)\cosh(-\mu a)) + \frac{i\alpha}{(-\mu)^2}(1 - \cos(-\mu a)\cosh(-\mu a)) \\
&= -i\alpha\mu^2 - i\alpha\mu^2\cos(\mu a)\cosh(\mu a) - \frac{-1}{\mu^3}(-\cos(\mu a)\sinh(\mu a) \\
&\quad + \sin(\mu a)\cosh(\mu a)) + \frac{i\alpha}{\mu^2}(1 - \cos(\mu a)\cosh(\mu a)) \\
&= -i\alpha\mu^2 - i\alpha\mu^2\cos(\mu a)\cosh(\mu a) - \frac{1}{\mu^3}(\cos(\mu a)\sinh(\mu a) \\
&\quad - \sin(\mu a)\cosh(\mu a)) + \frac{i\alpha}{\mu^2}(1 - \cos(\mu a)\cosh(\mu a)) = \phi_1(\mu).
\end{aligned} \tag{4.277}$$

Let

$$\begin{aligned}
\widetilde{\phi}_0(\mu) &= \alpha^2\mu^3(\cos(\mu a)\sinh(\mu a) + \sin(\mu a)\cosh(\mu a)) \\
&\quad - \frac{1}{\mu^3}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a))
\end{aligned} \tag{4.278}$$

$$\widetilde{\phi}_1(\mu) = -i\alpha\mu^2 - i\alpha\mu^2\cos(\mu a)\cosh(\mu a) + \frac{i\alpha}{\mu^2}(1 - \cos(\mu a)\cosh(\mu a)). \tag{4.279}$$

Then

$$\phi(\mu) = \widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu). \tag{4.280}$$

On the other hand

$$\begin{aligned}
\widetilde{\phi}_0(i\mu) &= \alpha^2(i\mu)^3(\cos(i\mu a)\sinh(i\mu a) + \sin(i\mu a)\cosh(i\mu a)) \\
&\quad - \frac{1}{(i\mu)^3}(\cos(i\mu a)\sinh(i\mu a) - \sin(i\mu a)\cosh(i\mu a)) \\
&= -i\alpha^2\mu^3(i\cosh(\mu a)\sin(\mu a) + i\sinh(\mu a)\cos(\mu a)) \\
&\quad - \frac{-1}{i\mu^3}(i\cosh(\mu a)\sin(\mu a) - i\sinh(\mu a)\cos(\mu a)) \\
&= \alpha^2\mu^3(\cos(\mu a)\sinh(\mu a) + \sin(\mu a)\cosh(\mu a)) \\
&\quad - \frac{1}{\mu^3}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) = \widetilde{\phi}_0(\mu),
\end{aligned} \tag{4.281}$$

while

$$\begin{aligned}
\widetilde{\phi}_1(i\mu) &= -i\alpha(i\mu)^2 - i\alpha(i\mu)^2\cos(i\mu a)\cosh(i\mu a) + \frac{i\alpha}{(i\mu)^2}(1 - \cos(i\mu a)\cosh(i\mu a)) \\
&= i\alpha\mu^2 + i\alpha\mu^2\cos(\mu a)\cosh(\mu a) - \frac{i\alpha}{\mu^2}(1 - \cos(\mu a)\cosh(\mu a)) = -\widetilde{\phi}_1(\mu).
\end{aligned} \tag{4.282}$$

Thus

$$\phi(i\mu) = \widetilde{\phi}_0(\mu) - \widetilde{\phi}_1(\mu). \tag{4.283}$$

It follows from (4.276) and (4.277) that $\phi_0$ and $\phi_1$ are even functions, whence $\phi$ is an even function. Therefore we have the same estimates (4.274) and (4.275) for all $\mu$ on the rectangles $R_k$ and $R_{-k}$ where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough. On the other hand, it follows from (4.280) and (4.283) that $|\phi(\mu)|$ and $|\phi(i\mu)|$ have the same upper bound $|\widetilde{\phi}_0(\mu)| + |\widetilde{\phi}_1(\mu)|$ for all $\mu$ on the rectangles $R_k$ and $R_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough. Hence we have the same estimates (4.274) and (4.275) all $\mu$ on the rectangles $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough.

Since 0 is a zero of multiplicity 4, $\hat{\mu}_k^{\pm}$, $\hat{\mu}_{-k}^{\pm}$, $k \in \mathbb{Z}$ are simple zeros of $\phi_0$, the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{Z}$ are closed curves, then it follows from (4.275), Remark 4.34 and Rouché's theorem that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$, where the asymptotics of the zeros of $\phi_0$ are

$$\left.\begin{aligned} \hat{\mu}_k^{\pm} &= \pm(4k-5)\tfrac{\pi}{4a} + o(1), \text{ where } \ k \geq k_0(\varepsilon) \\ \hat{\mu}_k^{\pm} &= \pm i(4|k|-5)\tfrac{\pi}{4a} + o(1), \text{ where } \ k \leq -k_0(\varepsilon) \end{aligned}\right\}, \tag{4.284}$$

with $o(1) \to 0$ as $|k| \to \infty$, see (4.270).

Let $S_k$, $k \in \mathbb{N}$ be the square defined in (4.66) and

$$\phi_{10}(\mu) = -i\alpha\cos(\mu a)\cosh(\mu a)\left(\mu^2 + \frac{1}{\mu^2}\right) \tag{4.285}$$

$$\phi_{11}(\mu) = -\frac{1}{\mu^3}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) \tag{4.286}$$

$$\phi_{12}(\mu) = i\alpha\left(-\mu^2 + \frac{1}{\mu^2}\right). \tag{4.287}$$

Let

$$\widetilde{\phi}_{10}(\mu) = \cos(\mu a)\cosh(\mu a). \tag{4.288}$$

For all $\mu$ on the square $S_k$, $k \in \mathbb{N}$, we have $\cos(\mu a)\cosh(\mu a) \neq 0$. Then it follows from (4.227) and (4.249) that there exists $\hat{k}_0 = \frac{13a}{\alpha\pi}$ such that for all $\mu$ on the square $S_k$ with vertices

$\pm k\frac{\pi}{a} \pm ik\frac{\pi}{a}$, where $k \in \mathbb{N}$, $k \geq \hat{m}_0 = \max\{\widetilde{k}_0, \hat{k}_0\}$,

$$\begin{aligned}\frac{\alpha^2}{3}|\phi_0(\mu)| &= \frac{\alpha^2}{3}|\mu|^3|\cos(\mu a)\sinh(\mu a) + \sin(\mu a)\cosh(\mu a)| \\ &= \frac{\alpha^2}{3}|\mu|^3|\cos(\mu a)\cosh(\mu a)||\tan(\mu a) + \tanh(\mu a)|\frac{2|\mu|^2}{2|\mu|^2} \\ &\geq \frac{\alpha|\mu|^3}{6|\mu|^2}|\widetilde{\psi}_1(\mu)|\alpha\left|\mu^2 + \frac{1}{\mu^2}\right||\cos(\mu a)\cosh(\mu a)| \\ &= \frac{\alpha|\mu|^3}{12|\mu|^2}|\phi_{10}(\mu)| = \frac{\alpha|\mu|}{12}|\phi_{10}(\mu)| > |\phi_{10}(\mu)| \end{aligned} \tag{4.289}$$

It follows from (4.258) that there exists $k_0 = \frac{5a}{\alpha\pi}$ and $\overline{k}_0 = \frac{a\ln 8}{\pi}$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$, $k \geq m_0 = \max\{k_0, \widetilde{k}_0, \overline{k}_0\}$

$$\begin{aligned}|\phi_{12}(\mu)| &= \alpha|\mu|^2\left|-1 + \frac{1}{\mu^4}\right| \leq 2\alpha|\mu|^2 \\ &\leq \frac{4}{\alpha|\mu|}\frac{\alpha^2}{3}|\tanh(\mu a) + \tan(\mu a)||\cosh(\mu a)\cos(\mu a)| < \frac{1}{3}\alpha^2|\phi_0(\mu)|. \end{aligned} \tag{4.290}$$

It follows from (4.227), (4.249) and (4.253) that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$, $k \geq \bar{m}_0 = \max\left\{\widetilde{k}_0, \frac{a}{\pi}\sqrt[6]{\frac{7M_0}{\alpha^2}}\right\}$

$$\begin{aligned}|\phi_{11}(\mu)| &= \frac{1}{|\mu|^3}|\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)| \\ &= \frac{1}{|\mu|^3}|\cos(\mu a)\cosh(\mu a)||\tan(\mu a) - \tanh(\mu a)| \\ &\leq \frac{2M_0}{2|\mu|^3}|\cos(\mu a)\cosh(\mu a)| \\ &\leq \frac{6M_0}{\alpha^2|\mu|^6}\frac{\alpha^2}{3}|\mu|^3|\cos(\mu a)\cosh(\mu a)||\tan(\mu a) + \tanh(\mu a)| \\ &\leq \frac{6M_0}{\alpha^2|\mu|^6}\frac{\alpha^2}{3}|\phi_0(\mu)| < \frac{\alpha^2}{3}|\phi_0(\mu)|. \end{aligned} \tag{4.291}$$

Putting together (4.289), (4.290) and (4.291), we have

$$|\phi_1(\mu)| < \alpha^2|\phi_0(\mu)|. \tag{4.292}$$

Let $\breve{m}_0 = \{m_0, \hat{m}_0, \bar{m}_0\}$. Then it follows from (4.292) and Rouché's theorem that $\phi_0$ and $\phi$ have the same number of zeros on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \breve{m}_0$.

**Proposition 4.38.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2)*, where $B_1(y) = y(0)$, $B_2(y) = y'(0)$, $B_3y = y''(a) + i\alpha\lambda y'(a)$*

*and $B_4 y = y(a) + i\alpha\lambda y^{(3)}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{cases} \hat{\mu}_k^{\pm} &= \pm(4k-5)\frac{\pi}{4a} + o(1), \quad if\ k > 0, \\ \hat{\mu}_k^{\pm} &= \pm i(4|k|-5)\frac{\pi}{4a} + o(1), \quad if\ k < 0. \end{cases}$$

*In particular, there is an even number of pure imaginary eigenvalues.*

**Remark 4.39.** We give the enumeration of the zeros of $\phi$ inside the square $S_{k_1}$, where $k_1 = \max\{k_0(\varepsilon), \breve{m}_0\}$. For the remainder of the proof of this proposition, we refer the reader to the remainder of the proof derived for Proposition 4.29.

*Proof.* We recall that the number of zeros of $\phi_0$ inside the square $S_{k_1}$ is $4k_1$, since 0 is a zero of multiplicity 4 of $\phi_0$, while $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$, $k \in \mathbb{N}$ are its simple zeros. Thus the number of zeros of the function $\phi$ inside the square $S_{k_1}$ is $4k_1$. As there is no nonzero of $\phi_0$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, the zeros of $\phi$ inside the square $S_{k_1}$ are the following

$$\hat{\mu}_1^{\pm} = \pm\left(-\frac{\pi}{4a}\right) + o(1),\ \hat{\mu}_2^{\pm} = \pm\left(\frac{3\pi}{4a}\right) + o(1), \dots, \hat{\mu}_{k_1} = \pm(4k_1-5)\frac{\pi}{4a} + o(1),$$
$$\hat{\mu}_{-1}^{\pm} = \pm i\left(-\frac{\pi}{4a}\right) + o(1),\ \hat{\mu}_{-2}^{\pm} = \pm i\left(\frac{3\pi}{4a}\right) + o(1) \dots, \hat{\mu}_{-(k_1)} = \pm i(4k_1-5)\frac{\pi}{4a} + o(1).$$

Thus

$$\hat{\mu}_k^{\pm} = \pm(4k-5)\frac{\pi}{4a} + o(1),\ \text{where } 1 \leq k \leq k_1 + 1,$$
$$\hat{\mu}_k^{\pm} = \pm i(4|k|-5)\frac{\pi}{4a} + o(1),\ \text{where } -k_1 - 1 \leq k \leq -1.$$

We can prove, using the approach of the proof of Proposition 4.23, that the zeros of $\phi$ for $|k| \geq k_1$ are

$$\hat{\mu}_k^{\pm},\ k = k_1, k_1+1, \dots$$
$$-k_1, -k_1-1, \dots$$

and satisfy

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(4|k|-5)\frac{\pi}{4a} + o(1),\ \text{where } k \geq k_1, \\ \hat{\mu}_k^{\pm} = \pm i(4|k|-5)\frac{\pi}{4a} + o(1),\ \text{where } k \leq -k_1, \end{cases}$$

see (4.284). $\square$

### 4.5.3 Asymptotic of the eigenvalues for $B_3y = y'(a) - i\alpha\lambda y''(a)$ and $B_4y = y^{(3)}(a) - i\alpha\lambda y(a)$

It follows from (4.207) that

$$\begin{aligned}
\det M &= B_3y_3B_4y_4 - B_4y_3B_3y_4 \\
&= (y_3'(a) - i\alpha\mu^2 y_3''(a))(y_4^{(3)}(a) - i\alpha\mu^2 y_4(a)) - (y_3^{(3)}(a) - i\alpha\mu^2 y_3(a))(y_4'(a) - i\alpha\mu^2 y_4''(a)) \\
&= y_3'(a)y_4^{(3)}(a) - \alpha^2\mu^4 y_3''(a)y_4(a) - y_3^{(3)}(a)y_4'(a) + \alpha^2\mu^4 y_3(a)y_4''(a) - i\alpha\mu^2(y_3''(a)y_4^{(3)}(a) \\
&+ y_3'(a)y_4(a) - y_3^{(3)}(a)y_4''(a) - y_3(a)y_4'(a)) \\
&= y_3'(a)y_4^{(3)}(a) - \alpha^2\mu^4 y_3''(a)y_4(a) - y_3^{(3)}(a)y_4'(a) + \alpha^2\mu^4 y_3(a)y_4''(a) \\
&- i\alpha\mu^2(y_3''(a)y_4^{(3)}(a) + y_3'(a)y_4(a) - y_3^{(3)}(a)y_4''(a) - y_3(a)y_4'(a)). \qquad (4.293)
\end{aligned}$$

Let

$$B_0 = y_3'(a)y_4^{(3)}(a) - \alpha^2\mu^4 y_3''(a)y_4(a) - y_3^{(3)}(a)y_4'(a) + \alpha^2\mu^4 y_3(a)y_4''(a) \qquad (4.294)$$

and

$$B_1 = y_3''(a)y_4^{(3)}(a) + y_3'(a)y_4(a) - y_3^{(3)}(a)y_4''(a) - y_3(a)y_4'(a). \qquad (4.295)$$

It follows from (4.209) that

$$B_0 = y_4''(a)y_4^{(3)}(a) - \alpha^2\mu^4 y_4^{(3)}(a)y_4(a) - \mu^4 y_4(a)y_4'(a) + \alpha^2\mu^4 y_4'(a)y_4''(a),$$

while

$$B_1 = (y_4^{(3)}(a))^2 + y_4''(a)y_4(a) - \mu^4 y_4(a)y_4''(a) - (y_4'(a))^2.$$

Using (4.29), (4.212), (4.213) and (4.214), we have

$$
\begin{aligned}
B_0 &= \frac{1}{4\mu}\sin(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sin(\mu a)\cosh(\mu a)\\
&\quad + \frac{1}{4\mu}\sinh(\mu a)\cosh(\mu a) - \alpha^2\mu^4\left(-\frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a)\right.\\
&\quad \left.-\frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a)\right) - \mu^4\left(\frac{1}{4\mu^5}\sin(\mu a)\cos(\mu a)\right.\\
&\quad \left.-\frac{1}{4\mu^5}\sin(\mu a)\cosh(\mu a) - \frac{1}{4\mu^5}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu^5}\sinh(\mu a)\cosh(\mu a)\right)\\
&\quad + \frac{\alpha^2\mu^4}{4\mu^3}(-\sin(\mu a)\cos(\mu a) - \sinh(\mu a)\cos(\mu a)\\
&\quad + \sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cosh(\mu a))\\
&= \frac{1}{2}\alpha^2\mu(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a))\\
&\quad + \frac{1}{2\mu}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)).
\end{aligned}
\tag{4.296}
$$

But (4.33), (4.211) and (4.31) give

$$
\begin{aligned}
B_1 &= \frac{1}{4}\cos^2(\mu a) + \frac{1}{2}\cos(\mu a)\cosh(\mu a) + \frac{1}{4}\cosh^2(\mu a)\\
&\quad - \frac{1}{4\mu^4}\sin^2(\mu a) + \frac{1}{4\mu^4}\sinh^2(\mu a) - \mu^4\left(-\frac{1}{4\mu^4}\sin^2(\mu a) + \frac{1}{4\mu^4}\sinh^2(\mu a)\right)\\
&\quad - \left(\frac{1}{4\mu^4}\cos^2(\mu a) - \frac{1}{2\mu^4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4\mu^4}\cosh^2(\mu a)\right)\\
&= \frac{1}{2}(1 + \cos(\mu a)\cosh(\mu a)) + \frac{1}{2\mu^4}(-1 + \cos(\mu a)\cosh(\mu a)).
\end{aligned}
\tag{4.297}
$$

It follows from (4.296) and (4.297) that

$$
\begin{aligned}
\det M &= \frac{1}{2}\alpha^2\mu(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a))\\
&\quad + \frac{1}{2\mu}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a))\\
&\quad - i\alpha\mu^2\left(\frac{1}{2}(1 + \cos(\mu a)\cosh(\mu a)) + \frac{1}{2\mu^4}(-1 + \cos(\mu a)\cosh(\mu a))\right)\\
&= -\frac{i\alpha\mu^2}{2}\cos(\mu a)\cosh(\mu a) - \frac{i\alpha\mu^2}{2}\\
&\quad + \frac{\alpha^2\mu}{2}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a))\\
&\quad + \frac{1}{2\mu}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a))\\
&\quad - \frac{i\alpha}{2\mu^2}\cos(\mu a)\cosh(\mu a) + \frac{i\alpha}{2\mu^2}.
\end{aligned}
\tag{4.298}
$$

Hence the characteristic equation $2i \det M = 0$ is

$$\phi_0(\mu) = \alpha\phi_0(\mu) + \phi_1(\mu) = 0, \tag{4.299}$$

where

$$\phi_0(\mu) = \mu^2 \cos(\mu a)\cosh(\mu a) \tag{4.300}$$

$$\begin{aligned}\phi_1(\mu) &= \alpha\mu^2 + i\alpha^2\mu(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \\ &\quad + \frac{i}{\mu}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) + \frac{\alpha}{\mu^2}\cos(\mu a)\cosh(\mu a) - \frac{\alpha}{\mu^2}.\end{aligned} \tag{4.301}$$

The function $\phi_0$ of this subsection is the half of the function $\phi_0$ defined in Subsection 4.4.1, see (4.40). Therefore the zeros of $\phi_0$ counted with multiplicity are:

$$\left.\mu_0^{0\pm} = 0,\ \mu_k^{0\pm} = \pm(2k-1)\tfrac{\pi}{2a}, \quad \text{and } \widetilde{\mu}_k^{0\pm} = \pm i(2k-1)\tfrac{\pi}{2a}, \text{ with } \quad k = 1, 2, \dots \right\}, \tag{4.302}$$

see (4.46).

Let $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ be the rectangles defined in (4.47) and

$$\phi_{11}(\mu) = \alpha\mu^2, \tag{4.303}$$

$$\phi_{12}(\mu) = i\alpha^2\mu(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)), \tag{4.304}$$

$$\phi_{13}(\mu) = \frac{i}{\mu}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)), \tag{4.305}$$

$$\phi_{14}(\mu) = \frac{\alpha}{\mu^2}\cos(\mu a)\cosh(\mu a), \tag{4.306}$$

$$\phi_{15}(\mu) = -\frac{\alpha}{\mu^2}. \tag{4.307}$$

Then

$$\phi_1(\mu) = \phi_{11}(\mu) + \phi_{12}(\mu) + \phi_{13}(\mu) + \phi_{14}(\mu) + \phi_{15}(\mu).$$

It follows from (4.258) that there exists $\hat{j}_0 = \frac{a}{\pi}\ln 12$ such that for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \hat{j}_0$, $|\cos(\mu a)\cosh(\mu a)| > 5$. Thus for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$ , where $k \in \mathbb{N}$ and $k \geq \hat{j}_0$,

$$\begin{aligned}\frac{\alpha}{5}|\phi_0(\mu)| &= \frac{\alpha}{5}|\mu|^2|\cos(\mu a)\cosh(\mu a)| \\ &> \alpha|\mu|^2 = |\phi_{11}(\mu)|.\end{aligned} \tag{4.308}$$

It follows from (4.48) and (4.50) that there exists $j_0$ such that for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$,where $k \in \mathbb{N}$ and $k \geq j_0$,

$$
\begin{aligned}
|\phi_{12}(\mu)| &= \alpha^2|\mu||\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)| \\
&\leq \frac{5\alpha(\beta_1+\beta_2)}{|\mu|}\frac{1}{5}\alpha|\mu|^2|\cos(\mu a)\cosh(\mu a)| \\
&= \frac{5\alpha(\beta_1+\beta_2)}{|\mu|}\frac{\alpha}{5}|\mu|^2|\cos(\mu a)\cosh(\mu a)| \qquad (4.309)
\end{aligned}
$$

and there exists $j_0$ such that for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$,where $k \in \mathbb{N}$ and $k \geq j_0$,

$$
\begin{aligned}
|\phi_{13}(\mu)| &= \frac{1}{|\mu|}|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\
&\leq \frac{1}{5}\frac{5\alpha}{\alpha|\mu|}(\beta_1+\beta_2)|\cos(\mu a)\cosh(\mu a)| \\
&= \frac{5(\beta_1+\beta_2)}{\alpha|\mu|^3}\frac{\alpha}{5}|\mu|^2|\cos(\mu a)\cosh(\mu a)|. \qquad (4.310)
\end{aligned}
$$

There exist $\widetilde{j}_0 = \frac{a}{\pi}\sqrt[4]{6}$ and $\bar{j}_0 = \frac{a}{\pi}$ positive such that for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \widetilde{j}_0$,

$$
\begin{aligned}
|\phi_{14}(\mu)| &= \frac{1}{5}\frac{5\alpha}{|\mu|^2}|\cos(\mu a)\cosh(\mu a)| \\
&= \frac{5}{|\mu|^4}\frac{\alpha}{5}|\mu|^2|\cos(\mu a)\cosh(\mu a)| = \frac{5}{|\mu|^4}\frac{\alpha}{5}|\phi_0(\mu)| \\
&< \frac{\alpha}{5}|\phi_0(\mu)| \qquad (4.311)
\end{aligned}
$$

and for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \bar{m}_0 = \max\{\hat{j}_0, \bar{j}_0\}$,

$$
\begin{aligned}
|\phi_{15}(\mu)| &= \frac{\alpha}{|\mu|^2} \\
&< \frac{1}{|\mu|^4}\frac{\alpha}{5}|\mu|^2|\cos(\mu a)\cosh(\mu a)| \\
&< \frac{\alpha}{5}|\mu|^2|\cos(\mu a)\cosh(\mu a)| = \frac{\alpha}{5}|\phi_0(\mu)|. \qquad (4.312)
\end{aligned}
$$

It follows from (4.309) that for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq m_0 = \max\left\{j_0, \frac{6\alpha(\beta_1+\beta_2)a}{\pi}\right\}$,

$$
|\phi_{12}(\mu)| < \frac{\alpha}{5}|\phi_0(\mu)|, \qquad (4.313)
$$

while (4.310) implies that for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_0 = \max\left\{ j_0, \frac{a}{\pi}\sqrt[3]{\frac{6(\beta_1+\beta_2)}{\alpha}} \right\}$,

$$|\phi_{13}(\mu)| < \frac{\alpha}{5}|\phi_0(\mu)|. \tag{4.314}$$

Hence putting (4.308), (4.311), (4.312), (4.313) and (4.314), we have for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_0 = \max\{\hat{j}_0, \tilde{j}_0, \bar{m}_0, m_0, \widetilde{m}_0\}$,

$$|\phi_1(\mu)| < \alpha|\phi_0(\mu)|. \tag{4.315}$$

Since the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{N}$ are closed curves, it follows from (4.315) and Rouché'theorem that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$, where the asymptotics of the zeros of $\phi_0$ are:

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(2k-1)\frac{\pi}{2a} + o(1) & \text{where } k \geq \hat{m}_0 \text{ and} \\ \hat{\mu}_k^{\pm} = \pm i(2|k|-1)\frac{\pi}{2a} + o(1), & \text{where } k \leq -\hat{m}_0, \end{cases}$$

with $o(1) \to 0$ as $|k| \to \infty$, see (4.65).

Let $S_k$, $k \in \mathbb{N}$ be the square defined in (4.66). There exists $\hat{j}_1 = \frac{1}{\pi}\ln 12$ positive such that for all $\mu$ on the square $S_k$ , where $k \in \mathbb{N}$ and $k \geq \hat{j}_1$, $|\cos(\mu a)\cosh(\mu a)| > 5$. Thus for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \hat{j}_1$,

$$\begin{aligned} \frac{\alpha}{5}|\phi_0(\mu)| &= \frac{\alpha}{5}|\mu|^2|\cos(\mu a)\cosh(\mu a)| \\ &> \alpha|\mu|^2 = |\phi_{11}(\mu)|. \end{aligned} \tag{4.316}$$

It follows from (4.68) and (4.69) that there exists $j_1 \in \mathbb{N}$ for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq j_1$

$$\begin{aligned} |\phi_{12}(\mu)| &= \alpha^2|\mu||\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)| \\ &\leq 5\alpha(\delta_1+\delta_2)\frac{1}{5}\alpha|\mu||\cos(\mu a)\cosh(\mu a)| \\ &= \frac{5\alpha(\delta_1+\delta_2)}{\alpha|\mu|^3}\frac{\alpha}{5}|\mu|^2|\cos(\mu a)\cosh(\mu a)| \end{aligned} \tag{4.317}$$

and

$$\begin{aligned}
|\phi_{13}(\mu)| &= \frac{1}{|\mu|}|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\
&\le \frac{1}{5}\frac{5\alpha}{\alpha|\mu|}(\delta_1 + \delta_2)|\cos(\mu a)\cosh(\mu a)| \\
&= \frac{5}{\alpha|\mu|^3}(\delta_1 + \delta_2)\frac{\alpha}{5}|\mu|^2|\cos(\mu a)\cosh(\mu a)|. \qquad (4.318)
\end{aligned}$$

Thus for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \ge m_1 = \max\left\{j_1, \frac{6\alpha(\delta_1+\delta_2)a}{\pi}\right\}$,

$$|\phi_{12}(\mu)| < \frac{\alpha}{5}|\phi_0(\mu)| \qquad (4.319)$$

and for all $\mu$ on the square $S_k$, $k \in \mathbb{N}$ and $k \ge \widetilde{m}_1 = \max\left\{j_1, \frac{a}{\pi}\sqrt[3]{\frac{6(\delta_1+\delta_2)}{\alpha}}\right\}$,

$$|\phi_{13}(\mu)| < \frac{\alpha}{5}|\phi_0(\mu)|. \qquad (4.320)$$

There exist $\widetilde{j}_1 = \frac{a}{\pi}\sqrt[4]{6}$ and $\bar{j}_1 = \frac{a}{\pi}$ positive such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \ge \widetilde{j}_1$,

$$\begin{aligned}
|\phi_{14}(\mu)| &= \frac{1}{5}\frac{5\alpha}{|\mu|^2}|\cos(\mu a)\cosh(\mu a)| \\
&= \frac{5}{|\mu|^4}\frac{\alpha}{5}|\mu|^2|\cos(\mu a)\cosh(\mu a)| \le \frac{\alpha}{5}|\phi_0(\mu)| \qquad (4.321)
\end{aligned}$$

and for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \ge \bar{m}_1 = \{\hat{j}_1, \bar{j}_1\}$,

$$\begin{aligned}
|\phi_{15}(\mu)| &= \frac{\alpha}{|\mu|^2} \\
&< \frac{1}{|\mu|^4}\frac{\alpha}{5}|\mu|^2|\cos(\mu a)\cosh(\mu a)| \\
&< \frac{\alpha}{5}|\mu|^2|\cos(\mu a)\cosh(\mu a)| = \frac{\alpha}{5}|\phi_0(\mu)|. \qquad (4.322)
\end{aligned}$$

Hence using (4.316), (4.319), (4.320), (4.321) and (4.322), we have for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \ge \hat{m}_1 = \max\{\hat{j}_1, \widetilde{j}_1, \bar{m}_1, m_1, \widetilde{m}_1\}$,

$$|\phi_1(\mu)| < \alpha|\phi_0(\mu)|. \qquad (4.323)$$

As the square $S_k$ is a closed curve, then (4.323) and Rouché's theorem imply that the functions $\phi_0$ and $\phi$ have the same number of zeros inside the square $S_k$, where $k \in \mathbb{N}$ and $k \ge \breve{m}_0$.

**Remark 4.40.** As the function $\phi_0$ defined in this subsection is the half of the function $\phi_0$ defined in Subsection 4.4.1, then we have the following proposition.

**Proposition 4.41.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y(0)$, $B_2(y) = y'(0)$, $B_3y = y'(a) - i\alpha\lambda y''(a)$ and $B_4y = y^{(3)}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{cases} \hat{\mu}_k^{\pm} &= \pm(2k-1)\frac{\pi}{2a} + o(1), \quad if\ k > 0, \\ \hat{\mu}_k^{\pm} &= \pm i(2|k|-1)\frac{\pi}{2a} + o(1), \quad if\ k < 0. \end{cases}$$

*In particular, there is an odd number of pure imaginary eigenvalues.*

**Remark 4.42.** The proof of the above proposition is identical to the proof of Proposition 4.23.

### 4.5.4 Asymptotic of the eigenvalues for $B_3y = y'(a) - i\alpha\lambda y''(a)$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a)$

It follows from (4.207) that

$$\begin{aligned}
\det M &= B_3y_3B_4y_4 - B_4y_3B_3y_4 \\
&= (y_3'(a) - i\alpha\mu^2 y_3''(a))(y_4(a) + i\alpha\mu^2 y_4^{(3)}(a)) - (y_3(a) + i\alpha\mu^2 y_3^{(3)}(a))(y_4'(a) - i\alpha\mu^2 y_4''(a)) \\
&= y_3'(a)y_4(a) + \alpha^2\mu^4 y_3''(a)y_4^{(3)}(a) - y_3(a)y_4'(a) - \alpha^2\mu^4 y_3^{(3)}(a)y_4''(a) + i\alpha\mu^2(y_3'(a)y_4^{(3)}(a) \\
&\quad - y_3''(a)y_4(a) + y_3(a)y_4''(a) - y_3^{(3)}(a)y_4'(a)) \\
&= y_3'(a)y_4(a) + \alpha^2\mu^4 y_3''(a)y_4^{(3)}(a) - y_3(a)y_4'(a) - \alpha^2\mu^4 y_3^{(3)}(a)y_4''(a) \\
&\quad + i\alpha\mu^2(y_3'(a)y_4^{(3)}(a) - y_3''(a)y_4(a) + y_3(a)y_4''(a) - y_3^{(3)}(a)y_4'(a)).
\end{aligned} \tag{4.324}$$

Let

$$B_0 = y_3'(a)y_4(a) + \alpha^2\mu^4 y_3''(a)y_4^{(3)}(a) - y_3(a)y_4'(a) - \alpha^2\mu^4 y_3^{(3)}(a)y_4''(a) \tag{4.325}$$

and

$$B_1 = y_3'(a)y_4^{(3)}(a) - y_3''(a)y_4(a) + y_3(a)y_4''(a) - y_3^{(3)}(a)y_4'(a). \tag{4.326}$$

Then it follows from (4.209) that

$$\begin{aligned}B_0 &= y_4''(a)y_4(a) + \alpha^2\mu^4(y_4^{(3)}(a))^2 - (y_4'(a))^2 - \alpha^2\mu^8 y_4(a)y_4''(a) \\ &= (1-\alpha^2\mu^8)y_4''(a)y_4(a) + \alpha^2\mu^4(y_4^{(3)}(a))^2 - (y_4'(a))^2 \end{aligned} \tag{4.327}$$

and

$$B_1 = y_4''(a)y_4^{(3)}(a) - y_4^{(3)}(a)y_4(a) + y_4'(a)y_4''(a) - \mu^4 y_4(a)y_4'(a). \tag{4.328}$$

Using (4.31), (4.33) and (4.211) we have

$$\begin{aligned}B_0 &= (1-\alpha^2\mu^8)\left(-\frac{1}{4\mu^4}\sin^2(\mu a) + \frac{1}{4\mu^4}\sinh^2(\mu a)\right) \\ &\quad + \frac{\alpha^2\mu^4}{4}\left(\cos^2(\mu a) + 2\cos(\mu a)\cosh(\mu a) + \cosh^2(\mu a)\right) \\ &= \frac{\alpha^2\mu^4}{2} - \frac{\alpha^2}{2\mu^4} + \frac{\alpha^2\mu^4}{2}\cos(\mu a)\cosh(\mu a) + \frac{\alpha^2}{2\mu^4}\cos(\mu a)\cosh(\mu a), \end{aligned} \tag{4.329}$$

while (4.29), (4.212), (4.213) and (4.214) give

$$\begin{aligned}B_1 &= \frac{1}{4\mu}\sin(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sin(\mu a)\cosh(\mu a) \\ &\quad + \frac{1}{4\mu}\sinh(\mu a)\cosh(\mu a) + \frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) \\ &\quad + \frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a) - \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a) - \frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) \\ &\quad - \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a) \\ &\quad - \frac{1}{4\mu}\cos(\mu a)\sin(\mu a) + \frac{1}{4\mu}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu}\cos(\mu a)\sinh(\mu a) \\ &\quad - \frac{1}{4\mu}\sinh(\mu a)\cosh(\mu a) \\ &= \frac{1}{2\mu}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \\ &\quad + \frac{1}{2\mu^3}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)). \end{aligned} \tag{4.330}$$

It follows from (4.324), (4.325), (4.326), (4.329) and (4.330) that

$$\begin{aligned}\det M &= \frac{\alpha^2\mu^4}{2}\cos(\mu a)\cosh(\mu a) + \frac{\alpha^2\mu^4}{2} \\ &\quad + \frac{i\alpha\mu}{2}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \\ &\quad - \frac{i\alpha}{2\mu}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)) \\ &\quad + \frac{\alpha^2}{2\mu^4}\cos(\mu a)\cosh(\mu a) - \frac{\alpha^2}{2\mu^4}. \end{aligned} \tag{4.331}$$

Thus the characteristic determinant $2\det M = 0$ is

$$\phi(\mu) = \alpha^2\phi_0(\mu) + \phi_1(\mu) = 0, \tag{4.332}$$

where

$$\phi_0(\mu) = \mu^4\cos(\mu a)\cosh(\mu a), \tag{4.333}$$

$$\begin{aligned}\phi_1(\mu) = {}& \alpha^2\mu^4 + i\alpha\mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a))\\ & - \frac{i\alpha}{\mu}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a))\\ & + \frac{\alpha^2}{\mu^4}\cos(\mu a)\cosh(\mu a) - \frac{\alpha^2}{\mu^4}.\end{aligned} \tag{4.334}$$

The function $\phi_0$ defined in this subsection is identical to the function $\phi_0$ defined in Subsection 4.4.2, see (4.89). Hence the zeros of $\phi_0$ are:

$$0, \mu_k^\pm = \pm(2k-3)\frac{\pi}{2a} \text{ and } \widetilde{\mu}_k^\pm = \pm i(2k-3)\frac{\pi}{2a}, \text{ with } k \in \mathbb{N},$$

see (4.89) and (4.333). We recall that 0 is a zero of multiplicity 4 of $\phi_0$ while $\mu_k^\pm = \pm(2k-3)\frac{\pi}{2a}$ and $\widetilde{\mu}_k^\pm = \pm i(2k-3)\frac{\pi}{2a}$ are its simple zeros. Therefore the zeros of $\phi_0$, counted with multiplicity, are

$$\left.\begin{array}{l}\mu_{-1}^\pm = 0,\ \mu_1^\pm = 0,\ \mu_k^\pm = \pm(2k-3)\frac{\pi}{2a}\\ \text{and } \widetilde{\mu}_k^\pm = \pm i(2k-3)\frac{\pi}{2a},\ \text{with } k = 2,3\ldots\end{array}\right\}. \tag{4.335}$$

Let $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ be the rectangles defined in (4.47). Let

$$\phi_{11}(\mu) = \alpha^2\mu^4, \tag{4.336}$$

$$\phi_{12}(\mu) = i\alpha\mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)), \tag{4.337}$$

$$\phi_{13}(\mu) = -\frac{i\alpha}{\mu}(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)), \tag{4.338}$$

$$\phi_{14}(\mu) = \frac{\alpha^2}{\mu^4}\cos(\mu a)\cosh(\mu a), \tag{4.339}$$

$$\phi_{15}(\mu) = -\frac{\alpha^2}{\mu^4}. \tag{4.340}$$

Then

$$\phi_1(\mu) = \phi_{11}(\mu) + \phi_{12}(\mu) + \phi_{13}(\mu) + \phi_{14}(\mu) + \phi_{15}(\mu). \tag{4.341}$$

It follows from (4.258) that there exists $k_0 = \frac{a}{\pi}\ln 12$ positive such that for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq k_0$, $|\cos(\mu a)\cosh(\mu a)| > 5$. Thus for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq k_0$,

$$\frac{\alpha^2}{5}|\phi_0(\mu)| = \frac{\alpha^2}{5}|\mu|^4|\cos(\mu a)\cosh(\mu a)| \\ > \alpha^2|\mu|^4 = |\phi_{11}(\mu)|. \tag{4.342}$$

It follows from (4.48) and (4.50) for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$,where $k \in \mathbb{N}$ and $k \geq j_0$,

$$\begin{aligned} |\phi_{12}(\mu)| &= \alpha|\mu||\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\ &\leq \frac{5|\mu|}{\alpha}(\beta_1+\beta_2)\frac{\alpha^2}{5}|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{5(\beta_1+\beta_2)}{\alpha|\mu|^3}\frac{\alpha^2}{5}|\mu|^4|\cos(\mu a)\cosh(\mu a)| \end{aligned} \tag{4.343}$$

and

$$\begin{aligned} |\phi_{13}(\mu)| &= \frac{\alpha}{|\mu|}|\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)| \\ &\leq \frac{5}{\alpha|\mu|}\frac{\alpha^2}{5}(\beta_1+\beta_2)|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{5(\beta_1+\beta_2)}{\alpha|\mu|^5}\frac{\alpha^2}{5}|\mu|^4|\cos(\mu a)\cosh(\mu a)|. \end{aligned} \tag{4.344}$$

Thus (4.343) implies for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k > m_0 = \left\{ j_0, \frac{a}{\pi}\sqrt[3]{\frac{6(\beta_1+\beta_2)}{\alpha}} \right\}$,

$$|\phi_{12}(\mu)| < \frac{\alpha^2}{5}|\phi_0(\mu)|, \tag{4.345}$$

while (4.344) gives for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k > \widetilde{m}_0 = \left\{ j_0, \frac{a}{\pi}\sqrt[5]{\frac{6(\beta_1+\beta_2)}{\alpha}} \right\}$

$$|\phi_{13}(\mu)| < \frac{\alpha^2}{5}|\phi_0(\mu)|. \tag{4.346}$$

There exist $\widetilde{k}_0 = \frac{a}{\pi}\sqrt[8]{6}$ and $\bar{k}_0 = \frac{a}{\pi}$ such that for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \widetilde{k}_0$,

$$\begin{aligned}
|\phi_{14}(\mu)| &= \frac{\alpha^2}{|\mu|^4}|\cos(\mu a)\cosh(\mu a)| \\
&= \frac{5}{|\mu|^8}\frac{\alpha^2}{5}|\mu|^4|\cos(\mu a)\cosh(\mu a)| \\
&= \frac{5}{|\mu|^8}\frac{\alpha^2}{5}|\phi_0(\mu)| \\
&< \frac{\alpha^2}{5}|\phi_0(\mu)| \qquad (4.347)
\end{aligned}$$

while for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \bar{m}_0 = \max\{\bar{k}_0, k_0\}$,

$$\begin{aligned}
|\phi_{15}(\mu)| &= \frac{\alpha^2}{|\mu|^4} \\
&\leq \frac{1}{|\mu|^8}\frac{\alpha^4}{5}|\mu|^4|\cos(\mu a)\cosh(\mu a)| \\
&< \frac{\alpha^2}{5}|\phi_0(\mu)|. \qquad (4.348)
\end{aligned}$$

Let $\hat{m}_0 = \max\{k_0, \widetilde{k}_0, m_0, \widetilde{m}_0, \bar{m}_0\}$. Then for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k > \hat{m}_0$, we have

$$|\phi_1(\mu)| < \alpha^2|\phi_0(\mu)|. \qquad (4.349)$$

The rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ are closed curves, since 0 is a zero of multiplicity 4 of $\phi_0$, while $\mu_k^\pm$ and $\widetilde{\mu}_k^\pm$ are its simple zeros, then (4.349) and Rouché's theorem imply that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi$, where the asymptotics of the zeros of $\phi$ are

$$\begin{cases} \hat{\mu}_k^\pm = \pm(2k-3)\frac{\pi}{2a} + o(1) \quad \text{where } k \geq \hat{m}_0 \text{ and} \\ \hat{\mu}_k^\pm = \pm i(2|k|-3)\frac{\pi}{2a} + o(1), \ \text{where } k \leq -\hat{m}_0, \end{cases}$$

with $o(1) \to 0$ as $|k| \to \infty$, see (4.102).

Let $S_k$ be the square defined in (4.66). It follows from (4.258) that there exists $k_1 = \frac{a}{\pi}\ln 12$ positive such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq k_1$, $|\cos(\mu a)\cosh(\mu a)| > 5$.

Thus for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq k_1$,

$$\frac{\alpha^2}{5}|\phi_0(\mu)| = \frac{\alpha^2}{5}|\mu|^4|\cos(\mu a)\cosh(\mu a)| > \alpha^2|\mu|^4 = |\phi_{11}(\mu)|. \tag{4.350}$$

It follows from (4.68) and (4.69) that there exists $j_1 \in \mathbb{N}$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq j_1$

$$\begin{aligned} |\phi_{12}(\mu)| &= \alpha|\mu||\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\ &\leq \frac{5|\mu|}{\alpha}(\delta_1 + \delta_2)\frac{\alpha^2}{5}|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{5(\delta_1 + \delta_2)}{\alpha|\mu|^3}\frac{\alpha^2}{5}|\mu|^4|\cos(\mu a)\cosh(\mu a)| \end{aligned} \tag{4.351}$$

and

$$\begin{aligned} |\phi_{13}(\mu)| &= \frac{\alpha}{|\mu|}|\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)| \\ &\leq \frac{\alpha^2}{5}\frac{5}{\alpha|\mu|}(\delta_1 + \delta_2)|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{5(\delta_1 + \delta_2)}{\alpha|\mu|^5}\frac{\alpha^2}{5}|\mu|^4|\cos(\mu a)\cosh(\mu a)|. \end{aligned} \tag{4.352}$$

Thus (4.351) implies for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k > m_1 = \left\{j_1, \frac{a}{\pi}\sqrt[3]{\frac{6(\delta_1+\delta_2)}{\alpha}}\right\}$,

$$|\phi_{12}(\mu)| < \frac{\alpha^2}{5}|\phi_0(\mu)|, \tag{4.353}$$

while (4.352) gives for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k > \widetilde{m}_1 = \left\{j_1, \frac{a}{\pi}\sqrt[5]{\frac{6(\delta_1+\delta_2)}{\alpha}}\right\}$,

$$|\phi_{13}(\mu)| < \frac{\alpha^2}{5}|\phi_0(\mu)|. \tag{4.354}$$

While there exist $\widetilde{k}_1 = \frac{a}{\pi}\sqrt[4]{6}$ and $\bar{k}_1 = \frac{a}{\pi}$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \widetilde{k}_1$,

$$|\phi_{14}(\mu)| < \frac{\alpha^2}{5}|\phi_0(\mu)| \tag{4.355}$$

and for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \bar{m}_1 = \{\bar{k}_1, k_1\}$,

$$|\phi_{15}(\mu)| < \frac{\alpha^2}{5}|\phi_0(\mu)|. \tag{4.356}$$

Let $\hat{m}_1 = \max\{k_1, \widetilde{k}_1, \bar{m}_1, m_1, \widetilde{m}_1\}$. Then for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k > \hat{m}_1$, we have

$$|\phi_1(\mu)| < \alpha^2 |\phi_0(\mu)|. \tag{4.357}$$

Since the square $S_k$, $k \in \mathbb{N}$ is a closed curve, then it follows from (4.357) and Rouché's theorem that the functions $\phi_0$ and $\phi$ have the same number of zeros in the square $S_k$, where $k \in \mathbb{N}$ and $k > \hat{m}_1$.

**Proposition 4.43.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2), *where $B_1(y) = y(0)$, $B_2(y) = y'(0)$, $B_3 y = y'(a) - i\alpha\lambda y''(a)$ and $B_4 y = y(a) + i\alpha\lambda y^{(3)}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(2k-3)\frac{\pi}{2a} + o(1), & if\ k > 0, \\ \hat{\mu}_k^{\pm} = \pm i(2|k|-3)\frac{\pi}{2a} + o(1), & if\ k < 0. \end{cases}$$

*In particular, there is an even number of pure imaginary eigenvalues.*

**Remark 4.44.** For the proof of the above proposition, we refer to the proof of Proposition 4.25, because the function $\phi_0$ defined in this subsection is the same as the function $\phi_0$ defined in Subsection 4.4.2.

## 4.6 The boundary terms $B_1 y$ and $B_2 y$ are the following: $B_1 y = y''(0)$ and $B_2 y = y^{(3)}(0)$

Using the canonical fundamental system, then

$$\begin{cases} B_1 y_1 = y_1''(0) = 0, \\ B_1 y_2 = y_2''(0) = 0, \\ B_1 y_3 = y_3''(0) = 1, \\ B_1 y_4 = y_4''(0) = 0, \end{cases} \qquad \begin{cases} B_2 y_1 = y_1^{(3)}(0) = 0, \\ B_2 y_2 = y_2^{(3)}(0) = 0, \\ B_2 y_3 = y_3^{(3)}(0) = 0, \\ B_2 y_4 = y_4^{(3)}(0) = 1. \end{cases}$$

It follows that the characteristic matrix of this particular boundary problem is

$$M_c = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ B_3y_1 & B_3y_2 & B_3y_3 & B_3y_4 \\ B_4y_1 & B_4y_2 & B_4y_3 & B_4y_4 \end{pmatrix}. \tag{4.358}$$

The determinant of the characteristic matrix $M_c$ gives the characteristic function of the differential equation (3.2). The shape of the matrix $M_c$ leads to a reduced characteristic matrix of the boundary value problem.

The reduced characteristic matrix of the boundary value problem is

$$M = \begin{pmatrix} B_3 \\ B_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} B_3y_1 & B_3y_2 \\ B_4y_1 & B_4y_2 \end{pmatrix}. \tag{4.359}$$

It is easy to check that $\det M_c = \det M$.

### 4.6.1 Asymptotic of the eigenvalues for $B_3y = y''(a) + i\alpha\lambda y'(a)$ and $B_4y = y^{(3)}(a) - i\alpha\lambda y(a)$

It follows from (4.359) that

$$\begin{aligned}
\det M &= B_3y_1B_4y_2 - B_4y_1B_3y_2 \\
&= (y_1''(a) + i\alpha\mu^2y_1'(a))(y_2^{(3)}(a) - i\alpha\mu^2y_2(a)) - (y_1^{(3)}(a) - i\alpha\mu^2y_1(a))(y_2''(a) + i\alpha\mu^2y_2'(a)) \\
&= y_1''(a)y_2^{(3)}(a) + \alpha^2\mu^4y_1'(a)y_2(a) - y_1^{(3)}(a)y_2''(a) - \alpha^2\mu^4y_1(a)y_2'(a) + i\alpha\mu^2(y_1'(a)y_2^{(3)}(a) \\
&\quad - y_1''(a)y_2(a) - y_1^{(3)}(a)y_2'(a) + y_1(a)y_2''(a)) \\
&= y_1''(a)y_2^{(3)}(a) - y_1^{(3)}(a)y_2''(a) + \alpha^2\mu^4y_1'(a)y_2(a) - \alpha^2\mu^4y_1(a)y_2'(a) \\
&\quad + i\alpha\mu^2(y_1'(a)y_2^{(3)}(a) - y_1''(a)y_2(a) + y_1(a)y_2''(a) - y_2'(a)y_1^{(3)}(a)).
\end{aligned} \tag{4.360}$$

It follows from (4.21) and (4.25) that

$$\begin{cases} y_1(x) = y_4^{(3)}(x), \\ y_1'(x) = -\frac{\mu}{2}\sin(\mu a) + \frac{\mu}{2}\sinh(\mu a) = \mu^4 y_4(x), \\ y_1''(x) = -\frac{\mu^2}{2}\cos(\mu a) + \frac{\mu^2}{2}\cosh(\mu a) = \mu^4 y_4'(x), \\ y_1^{(3)}(x) = \frac{\mu^3}{2}\sin(\mu a) + \frac{\mu^3}{2}\sinh(\mu a) = \mu^4 y_4''(x), \end{cases} \tag{4.361}$$

while

$$\begin{cases} y_2(x) = \frac{1}{2\mu}\sin(\mu a) + \frac{1}{2\mu}\sinh(\mu a) = y_4''(x), \\ y_2'(x) = \frac{1}{2}\cos(\mu a) + \frac{1}{2}\cosh(\mu a) = y_4^{(3)}(x), \\ y_2''(x) = -\frac{\mu}{2}\sin(\mu a) + \frac{\mu}{2}\sinh(\mu a) = \mu^4 y_4(x), \\ y_2^{(3)}(x) = -\frac{\mu^2}{2}\cos(\mu a) + \frac{\mu^2}{2}\cosh(\mu a) = \mu^4 y_4'(x). \end{cases} \tag{4.362}$$

Thus

$$\begin{aligned} \det M &= \mu^8(y_4'(a))^2 - \mu^8 y_4''(a)y_4(a) + \alpha^2\mu^8 y_4(a)y_4''(a) - \alpha^2\mu^4(y_4^{(3)}(a))^2 \\ &\quad + i\alpha\mu^2(\mu^8 y_4(a)y_4'(a) - \mu^4 y_4'(a)y_4''(a) + \mu^4 y_4^{(3)}(a)y_4(a) - \mu^4 y_4^{(3)}(a)y_4''(a)) \\ &= \mu^8(y_4'(a))^2 + (\alpha^2-1)\mu^8 y_4(a)y_4''(a) - \alpha^2\mu^4(y_4^{(3)}(a))^2 \\ &\quad + i\alpha\mu^2(\mu^8 y_4(a)y_4'(a) - \mu^4 y_4'(a)y_4''(a) + \mu^4 y_4^{(3)}(a)y_4(a) - \mu^4 y_4^{(3)}(a)y_4''(a)). \end{aligned} \tag{4.363}$$

Let

$$A_1(a) = \mu^8(y_4'(a))^2 + (\alpha^2-1)\mu^8 y_4(a)y_4''(a) - \alpha^2\mu^4(y_4^{(3)}(a))^2 \text{ and} \tag{4.364}$$

$$A_2(a) = \mu^8 y_4(a)y_4'(a) - \mu^4 y_4'(a)y_4''(a) + \mu^4 y_4^{(3)}(a)y_4(a) - \mu^4 y_4^{(3)}(a)y_4''(a). \tag{4.365}$$

Then it follows from (4.31), (4.33) and (4.211) that

$$\begin{aligned} A_1(a) &= \mu^8\left(\frac{1}{4\mu^4}\cos^2(\mu a) - \frac{1}{2\mu^4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4\mu^4}\cosh^2(\mu a)\right) \\ &\quad + (\alpha^2-1)\mu^8\left(-\frac{1}{4\mu^4}\sin^2(\mu a) + \frac{1}{4\mu^4}\sinh^2(\mu a)\right) \\ &\quad - \alpha^2\mu^4\left(\frac{1}{4}\cos^2(\mu a) + \frac{2}{4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4}\cosh^2(\mu a)\right) \\ &= (1-\alpha^2)\frac{\mu^4}{2} - (1+\alpha^2)\frac{\mu^4}{2}\cos(\mu a)\cosh(\mu a), \end{aligned} \tag{4.366}$$

while (4.28), (4.29), (4.212) and (4.213) give

$$\begin{aligned}
A_2(a) = \mu^8 &\left(\frac{1}{4\mu^5}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^5}\sin(\mu a)\cosh(\mu a) - \frac{1}{4\mu^5}\sinh(\mu a)\cos(\mu a)\right. \\
&\left. + \frac{1}{4\mu^5}\sinh(\mu a)\cosh(\mu a)\right) - \mu^4\left(-\frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a)\right. \\
&\left. +\frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a)\right) + \mu^4\left(-\frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a)\right. \\
&\left. -\frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a)\right) \\
&- \mu^4\left(\frac{1}{4\mu}\sin(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sin(\mu a)\cosh(\mu a)\right. \\
&\left. +\frac{1}{4\mu}\sinh(\mu a)\cosh(\mu a)\right) \\
&= -\frac{\mu^3}{2}(\sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cos(\mu a)) + \frac{\mu}{2}(\sinh(\mu a)\cos(\mu a) \\
&- \sin(\mu a)\cosh(\mu a)).
\end{aligned} \tag{4.367}$$

Hence it follows from (4.363), (4.364), (4.365), (4.366) and (4.367) that

$$\begin{aligned}
\det M = (1-\alpha^2)\frac{\mu^4}{2} &- (1+\alpha^2)\frac{\mu^4}{2}\cos(\mu a)\cosh(\mu a) + i\alpha\mu^2(-\frac{\mu^3}{2}(\sin(\mu a)\cosh(\mu a) \\
&+ \sinh(\mu a)\cos(\mu a)) + \frac{\mu}{2}(\sinh(\mu a)\cos(\mu a) - \sin(\mu a)\cosh(\mu a)) \\
&= -\frac{i\alpha\mu^5}{2}(\sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cos(\mu a)) - (1+\alpha^2)\frac{\mu^4}{2}\cos(\mu a)\cosh(\mu a) \\
&+ (1-\alpha^2)\frac{\mu^4}{2} + \frac{i\alpha\mu^3}{2}(\sinh(\mu a)\cos(\mu a) - \sin(\mu a)\cosh(\mu a)).
\end{aligned} \tag{4.368}$$

Therefore the characteristic equation $2i\det M = 0$ is

$$\phi(\mu) := \alpha\phi_0(\mu) + \phi_1(\mu) = 0, \tag{4.369}$$

where

$$\phi_0(\mu) = \mu^5(\sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cos(\mu a)), \tag{4.370}$$

$$\begin{aligned}
\phi_1(\mu) = &-i(1+\alpha^2)\mu^4\cos(\mu a)\cosh(\mu a) + i(1-\alpha^2)\mu^4 \\
&- \alpha\mu^3(\sinh(\mu a)\cos(\mu a) - \sin(\mu a)\cosh(\mu a)).
\end{aligned} \tag{4.371}$$

The function $\phi_0$ defined in this subsection has the same zeros as the function $\phi_0$ defined in (4.222). However, because of their multiplicity, the zeros of $\phi_0$ defined in (4.370) will be given in the following form

$$0\ , \hat{\mu}_k^{\pm} = \pm(4k-5)\frac{\pi}{4a} + o(1),\ \hat{\mu}_{-k}^{\pm} = \pm i(4k-5)\frac{\pi}{4a} + o(1),\ k = 2,3,4,\dots. \tag{4.372}$$

The function $\phi_0$ in this subsection is obtained by multiplying the function $\phi_0$ in Subsection 4.4.3 by $\mu^2$ and by replacing the trigonometric functions $\sin(\mu a)$ and $\cos(\mu a)$ respectively with $\sin(-\mu a)$ and $\cos(-\mu a)$. This leads to the equation (4.128) for $\mu \neq 0$. Thus the zeros of $\phi_0$ are either real or pure imaginary, see Remark 4.26.

We recall that 0 is a zero of multiplicity 6 of the function $\phi_0$ since 0 is a zero of multiplicity 1 for the function $\widetilde{\psi}_1(\mu) = \tan(\mu a) + \tanh(\mu a)$ defined in (4.227) and it is a zero of multiplicity 5 for the function $\mu \mapsto \psi_0(\mu) = \mu^5$. Whence the zeros of $\phi_0$ counted with multiplicity are the following

$$\left.\begin{array}{l} \hat{\mu}_{-1}^{\pm} = 0,\ \hat{\mu}_0^{\pm} = 0,\ \hat{\mu}_1^{\pm} = 0,\ \hat{\mu}_k^{\pm} = \pm(4k-5)\frac{\pi}{4a} + o(1), \\ \hat{\mu}_{-k}^{\pm} = \pm i(4k-5)\frac{\pi}{4a} + o(1),\ k = 2,3,\dots \end{array}\right\}. \tag{4.373}$$

Let

$$\phi_{00}(\mu) = \cos(\mu a) + \sin(\mu a) \tag{4.374}$$

$$\begin{aligned}\phi_{01}(\mu) &= (-1+\tanh(\mu a))\cos(\mu a) + \frac{i(1-\alpha^2)}{\alpha\mu\cosh(\mu a)} - \frac{i(1+\alpha^2)}{\alpha\mu}\cos(\mu a) \\ &\quad + \frac{1}{\mu^2}(\sin(\mu a) - \cos(\mu a)\tanh(\mu a)).\end{aligned} \tag{4.375}$$

Then

$$\phi_{02}(\mu) = \frac{\phi(\mu)}{\alpha\mu^5\cosh(\mu a)} = \phi_{00}(\mu) + \phi_{01}(\mu). \tag{4.376}$$

We recall that

$$\mu_k^{00} = \left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right),\ k \in \mathbb{Z} \text{ are the zeros of } \phi_{00},$$

see (4.232) and

$$\widetilde{\mu}_k^{00} = i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right),\ k \in \mathbb{Z} \text{ are the images of } \mu_k^{00} \text{ by the rotation of angle } \frac{\pi}{2},$$

see (4.233).

Let $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ be the rectangles defined in (4.234). We recall that $\mu_k^{00} = \left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in R_k$, $-\mu_k^{00} = -\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in R_{-k}$, $\widetilde{\mu}_k^{00} = i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_k$ and $-\widetilde{\mu}_k^{00} = -i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_{-k}$, see Remark 4.34. The rectangles $R_k$ do not intersect, as well as the rectangles $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{Z}$, due to $\varepsilon < \frac{\pi}{2a}$. We recall also that there exists a constant $\rho(\varepsilon) > 0$ such that $|\phi_{00}(\mu)| > \rho(\varepsilon)$ for all $\mu$ on the rectangle $R_k$, $k \in \mathbb{Z}$, as $|\phi_{00}|$ is periodic of period $\frac{\pi}{a}$.

It follows from (4.235) and (4.236) that for all $\mu$ on the rectangle $R_k$, where $|k| > k_0(\varepsilon) = \max\{k_1(\varepsilon), k_2(\varepsilon)\}$ is sufficiently large positive, we have

$$|\phi_{01}(\mu)| < \frac{(3+\sqrt{2})(1+\alpha^2)}{\alpha|\mu|} + \frac{3\sqrt{2}}{|\mu|^2} + 3e^{-|\Re\mu a|}. \tag{4.377}$$

Since the right hand tends to 0 as $|\Re\mu a| \to \infty$, then for all $\mu$ on the rectangle $R_k$, where $k \in \mathbb{Z}$, $|k| > k_0(\varepsilon)$,

$$|\phi_{01}(\mu)| < |\phi_{00}(\mu)|. \tag{4.378}$$

For $\mu \in \mathbb{C}$, we have

$$\begin{aligned}
\phi_0(-\mu) &= (-\mu)^5(\sin(-\mu a)\cosh(-\mu a) + \sinh(-\mu a)\cos(-\mu a)) \\
&= -\mu^5(-\sin(\mu a)\cosh(\mu a) - \sinh(\mu a)\cos(\mu a)) \\
&= \mu^5(\sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cos(\mu a)) = \phi_0(\mu),
\end{aligned} \tag{4.379}$$

while

$$\begin{aligned}
\phi_1(-\mu) &= -i(1+\alpha^2)(-\mu)^4\cos(-\mu a)\cosh(-\mu a) + i(1-\alpha^2)(-\mu)^4 \\
&\quad - \alpha(-\mu)^3(\sinh(-\mu a)\cos(-\mu a) - \sin(-\mu a)\cosh(-\mu a)) \\
&= -i(1+\alpha^2)\mu^4\cos(\mu a)\cosh(\mu a) + i(1-\alpha^2)\mu^4 \\
&\quad + \alpha\mu^3(-\sinh(\mu a)\cos(\mu a) + \sin(\mu a)\cosh(\mu a)) \\
&= -i(1+\alpha^2)\mu^4\cos(\mu a)\cosh(\mu a) + i(1-\alpha^2)\mu^4 \\
&\quad - \alpha\mu^3(\sinh(\mu a)\cos(\mu a) - \sin(\mu a)\cosh(\mu a)) = \phi_1(\mu).
\end{aligned} \tag{4.380}$$

Thus the function $\phi_0$ and $\phi_1$ are even function and therefore $\phi$ is an even function. Hence we have the same estimates (4.377) and (4.378) for all $\mu$ on the squares $R_k$ and $R_{-k}$, where $k \in \mathbb{Z}$, $|k| > k_0(\varepsilon)$ is large enough.

Let

$$\widetilde{\phi}_0(\mu) = \alpha\mu^5(\sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cos(\mu a)) - \alpha\mu^3(\sinh(\mu a)\cos(\mu a) - \sin(\mu a)\cosh(\mu a)) \text{ and} \tag{4.381}$$

$$\widetilde{\phi}_1(\mu) = -i(1+\alpha^2)\mu^4\cos(\mu a)\cosh(\mu a) + i(1-\alpha^2)\mu^4. \tag{4.382}$$

Then

$$\phi(\mu) = \widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu). \tag{4.383}$$

On the other hand, we have

$$\begin{aligned}\widetilde{\phi}_0(i\mu) &= \alpha(i\mu)^5(\sin(i\mu a)\cosh(i\mu a) + \sinh(i\mu a)\cos(i\mu a))\\ &\quad - \alpha(i\mu)^3(\sinh(i\mu a)\cos(i\mu a) - \sin(i\mu a)\cosh(i\mu a))\\ &= i\alpha\mu^5(i\sinh(\mu a)\cos(\mu a) + i\sin(\mu a)\cosh(\mu a))\\ &\quad - \alpha(-i)\mu^3(i\sin(\mu a)\cosh(\mu a) - i\sinh(\mu a)\cos(\mu a))\\ &= -\alpha\mu^5(\sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cos(\mu a))\\ &\quad + \alpha\mu^3(\sinh(\mu a)\cos(\mu a) - \sin(\mu a)\cosh(\mu a)) = -\widetilde{\phi}_0(\mu),\end{aligned} \tag{4.384}$$

while

$$\begin{aligned}\widetilde{\phi}_1(i\mu) &= -i(1+\alpha^2)(i\mu)^4\cos(i\mu a)\cosh(i\mu a) + i(1-\alpha^2)(i\mu)^4\\ &= -i(1+\alpha^2)\mu^4\cos(\mu a)\cosh(\mu a) + i(1-\alpha^2)\mu^4 = \widetilde{\phi}_1(\mu).\end{aligned} \tag{4.385}$$

Thus

$$\phi(i\mu) = -\widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu). \tag{4.386}$$

Using (4.369), (4.370) and (4.371), we can obtain the same estimates (4.377) and (4.378) for all $\mu$ on the rectangles $R_k$ and $R_{-k}$ where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough. Since $|\phi(\mu)|$ and $|\phi(i\mu)|$ have the same upper bound $|\phi_0(\mu)| + |\phi_1(\mu)|$ for all $\mu$ on the rectangles $R_k$ and $R_{-k}$, see (4.383) and (4.386), then we can obtain the same estimates (4.377) and (4.378) for all $\mu$ on the rectangle $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough.

Since the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ are closed curves, 0 is a zero of multiplicity 6, $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$ are simple zeros of $\phi_0$ and $\phi_0$ has no nonzero zero in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, then it follows from (4.378), Remark 4.34 and Rouché's theorem that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$, where the asymptotics of the zeros of $\phi_0$ are

$$\left.\begin{aligned} \hat{\mu}_k^{\pm} &= \pm(4k-5)\tfrac{\pi}{4a} + o(1), \text{ where } k \in \mathbb{Z},\ k \geq k_0(\varepsilon) \text{ and} \\ \hat{\mu}_k^{\pm} &= \pm i(4|k|-5)\tfrac{\pi}{4a} + o(1), \text{ where } k \in \mathbb{Z},\ k \leq -k_0(\varepsilon) \end{aligned}\right\}, \tag{4.387}$$

with $o(1) \to 0$ as $|k| \to \infty$, see (4.373).

Let $S_k$, $k \in \mathbb{N}$ be the square defined in (4.66) and

$$\phi_{11}(\mu) = -i(1+\alpha^2)\mu^4 \cos(\mu a)\cosh(\mu a) \tag{4.388}$$

$$\phi_{12}(\mu) = -(1-\alpha^2)\mu^4 \tag{4.389}$$

$$\phi_{13}(\mu) = -\alpha\mu^3(\sinh(\mu a)\cos(\mu a) - \sin(\mu a)\cosh(\mu a)). \tag{4.390}$$

For all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \hat{k}_1 = \frac{7a(1+\alpha^2)}{\alpha\pi}$, we have

$$\begin{aligned} |\phi_{11}(\mu)| &= \frac{\alpha}{3}|\mu|^4|\cos(\mu a)\cosh(\mu a)|\left|\frac{3(1+\alpha^2)}{\alpha}\right| \\ &= \frac{6(1+\alpha^2)}{2\alpha|\mu|}\frac{\alpha}{3}|\mu|^5|\cos(\mu a)\cosh(\mu a)| \\ &\leq \frac{6(1+\alpha^2)}{\alpha|\mu|}\frac{\alpha}{3}|\mu|^5|\cos(\mu a)\cosh(\mu a)||\tan(\mu a)+\tanh(\mu a)| \\ &< \frac{\alpha}{3}|\mu|^5|\cos(\mu a)\cosh(\mu a)||\tan(\mu a)+\tanh(\mu a)| = \frac{\alpha}{3}|\phi_0(\mu)|, \end{aligned} \tag{4.391}$$

while there exists $\bar{k}_1 = \frac{a}{\pi}\ln 8$, such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \bar{m}_1 = \{\bar{k}_1, \frac{2a}{\pi}\frac{1+\alpha^2}{\alpha}\}$

$$\begin{aligned} |\phi_{12}(\mu)| &< \frac{\alpha(1+\alpha^2)}{\alpha}|\mu|^4 \\ &< \frac{\alpha(1+\alpha^2)}{3\alpha}|\mu|^4|\cos(\mu a)\cosh(\mu a)| \\ &< \frac{2(1+\alpha^2)}{\alpha|\mu|}\frac{\alpha}{3}|\mu|^5|\cos(\mu a)\cosh(\mu a)||\tan(\mu a)+\tanh(\mu a)| \\ &< \frac{\alpha}{3}|\mu|^5|\cos(\mu a)\cosh(\mu a)||\tan(\mu a)+\tanh(\mu a)| = \frac{\alpha}{3}|\phi_0(\mu)|. \end{aligned} \tag{4.392}$$

On the other hand, it follows from (4.253) that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and

$k \geq \widetilde{k}_1 = \frac{a}{\pi}\sqrt{7M_0}$

$$\begin{aligned}|\phi_{13}(\mu)| &= \frac{\alpha}{3}3|\mu|^3|\cos(\mu a)\cosh(\mu a)||\tan(\mu a) - \tanh(\mu a)| \\ &\leq \frac{3M_0\alpha}{3}|\mu|^3|\cos(\mu a)\cosh(\mu a)| \\ &\leq \frac{6M_0}{|\mu|^2}\frac{\alpha}{3}|\mu|^5|\cos(\mu a)\cosh(\mu a)||\tan(\mu a) + \tanh(\mu a)| < \frac{\alpha}{3}|\phi_0(\mu)|. \end{aligned} \tag{4.393}$$

It follows from (4.391), (4.392) and (4.393) that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \breve{k}_1 = \max\{\hat{k}_1, \widetilde{k}_1, \bar{m}_1\}$, we have

$$|\phi_1(\mu)| \leq |\phi_{11}(\mu)| + |\phi_{12}(\mu)| + |\phi_{13}(\mu)| < \alpha|\phi_0(\mu)|. \tag{4.394}$$

As the square $S_k$, $k \in \mathbb{N}$ is a closed curve, then it follows from (4.394) and Rouché's theorem that the functions $\phi_0$ and $\phi$ have the same number of zeros in the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \breve{k}_1$.

**Remark 4.45.** Let $k_1 = \max\{k_0(\varepsilon), \breve{k}_1\}$. As 0 is a zero of multiplicity 6 of $\phi_0$, while $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$, where $k = 2, 3, 4 \ldots$, are its simple zeros, then the number of zeros of $\phi_0$ and therefore of $\phi$ inside the square $S_k$, where $k \in \mathbb{N}$ and $k \geq k_1$ is $4k + 2$. Thus we have the following proposition.

**Proposition 4.46.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y''(0)$, $B_2(y) = y^{(3)}(0)$, $B_3 y = y''(a) + i\alpha\lambda y'(a)$ and $B_4 y = y^{(3)}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{cases} \hat{\mu}_k^{\pm} &= \pm(4k-5)\frac{\pi}{4a} + o(1), \quad if\ k > 0, \\ \hat{\mu}_k^{\pm} &= \pm i(4|k|-5)\frac{\pi}{4a} + o(1), \quad if\ k < 0. \end{cases}$$

*In particular, there is an odd number of pure imaginary eigenvalues.*

**Remark 4.47.** For the proof of the above proposition, we refer to the proof of Proposition 4.38.

### 4.6.2 Asymptotic of the eigenvalues for $B_3y = y''(a) + i\alpha\lambda y'(a)$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a)$

It follows from (4.207) that

$$\begin{aligned}
\det M &= B_3y_1B_4y_2 - B_4y_1B_3y_2\\
&= (y_1''(a) + i\alpha\mu^2y_1'(a))(y_2(a) + i\alpha\mu^2y_2^{(3)}(a)) - (y_1(a) + i\alpha\mu^2y_1^{(3)}(a))(y_2''(a) + i\alpha\mu^2y_2'(a))\\
&= y_1''(a)y_2(a) - \alpha^2\mu^4y_1'(a)y_2^{(3)}(a) - y_1(a)y_2''(a) + \alpha^2\mu^4y_1^{(3)}(a)y_2'(a) + i\alpha\mu^2(y_1'(a)y_2(a)\\
&+ y_1''(a)y_2^{(3)}(a) - y_1(a)y_2'(a) - y_1^{(3)}(a)y_2''(a))\\
&= y_1''(a)y_2(a) - y_1(a)y_2''(a) - \alpha^2\mu^4y_1'(a)y_2^{(3)}(a) + \alpha^2\mu^4y_1^{(3)}(a)y_2'(a)\\
&+ i\alpha\mu^2(y_1'(a)y_2(a) + y_1''(a)y_2^{(3)}(a) - y_1^{(3)}(a)y_2''(a) - y_2'(a)y_1(a)).
\end{aligned} \tag{4.395}$$

It follows from (4.361) and (4.362) that

$$\begin{aligned}
\det M &= \mu^4y_4'(a)y_4''(a) - \mu^4y_4^{(3)}(a)y_4(a) - \alpha^2\mu^{12}y_4(a)y_4'(a) + \alpha^2\mu^8y_4''(a)y_4^{(3)}(a)\\
&+ i\alpha\mu^2(\mu^4y_4(a)y_4''(a) + \mu^8(y_4'(a))^2 - \mu^8y_4''(a)y_4(a) - (y_4^{(3)}(a))^2).
\end{aligned} \tag{4.396}$$

Let

$$A_1(a) = \mu^4y_4'(a)y_4''(a) - \mu^4y_4^{(3)}(a)y_4(a) - \alpha^2\mu^{12}y_4(a)y_4'(a) + \alpha^2\mu^8y_4''(a)y_4^{(3)}(a) \text{ and} \tag{4.397}$$

$$A_2(\mu) = \mu^4y_4(a)y_4''(a) + \mu^8(y_4'(a))^2 - \mu^8y_4''(a)y_4(a) - (y_4^{(3)}(a))^2. \tag{4.398}$$

Then it follows from (4.28), (4.29), (4.212) and (4.213) that

$$\begin{aligned}
A_1(a) = {}& \mu^4\left(-\frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a)\right.\\
&\left. + \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a)\right) - \mu^4\left(-\frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a)\right.\\
&\left.+\frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a)\right) - \alpha^2\mu^{12}\left(\frac{1}{4\mu^5}\sin(\mu a)\cos(\mu a)\right.\\
&\left.-\frac{1}{4\mu^5}\sin(\mu a)\cosh(\mu a) - \frac{1}{4\mu^5}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu^5}\sinh(\mu a)\cosh(\mu a)\right)\\
& + \alpha^2\mu^8\left(\frac{1}{4\mu}\sin(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sin(\mu a)\cosh(\mu a)\right.\\
&\left.+\frac{1}{4\mu}\sinh(\mu a)\cosh(\mu a)\right)\\
={}& \frac{\mu}{2}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a))\\
&+ \frac{\alpha^2\mu^7}{2}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)),
\end{aligned} \tag{4.399}$$

while it follows from (4.31), (4.33) and (4.211)

$$\begin{aligned}
A_2(a) = {}& \mu^8\left(\frac{1}{4\mu^4}\cos^2(\mu a) - \frac{1}{2\mu^4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4\mu^4}\cosh^2(\mu a)\right)\\
&- \frac{1}{4}\cos^2(\mu a) - \frac{2}{4}\cos(\mu a)\cosh(\mu a) - \frac{1}{4}\cosh^2(\mu a)\\
&+ \mu^4\left(-\frac{1}{4\mu^4}\sin^2(\mu a) + \frac{1}{4\mu^4}\sinh^2(\mu a)\right) - \mu^8\left(-\frac{1}{4\mu^4}\sin^2(\mu a) + \frac{1}{4\mu^4}\sinh^2(\mu a)\right)\\
={}& -\frac{\mu^4+1}{2}\cos(\mu a)\cosh(\mu a) + \frac{\mu^4-1}{2}.
\end{aligned} \tag{4.400}$$

Using (4.396), (4.399) and (4.400), we get

$$\begin{aligned}
\det M = {}& \frac{\alpha^2\mu^7}{2}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) + \frac{\mu}{2}(\sin(\mu a)\cosh(\mu a)\\
&- \cos(\mu a)\sinh(\mu a)) - \frac{i\alpha(\mu^6+\mu^2)}{2}\cos(\mu a)\cosh(\mu a) + \frac{i\alpha(\mu^6-\mu^2)}{2}.
\end{aligned} \tag{4.401}$$

Thus the characteristic equation $2\det M = 0$ is

$$\phi(\mu) := \alpha^2\phi_0(\mu) + \phi_1(\mu) = 0, \tag{4.402}$$

where

$$\phi_0(\mu) = \mu^7(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)), \tag{4.403}$$

$$\begin{aligned}
\phi_1(\mu) = {}& \mu(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a))\\
&- i\alpha(\mu^6+\mu^2)\cos(\mu a)\cosh(\mu a) + i\alpha(\mu^6-\mu^2).
\end{aligned} \tag{4.404}$$

The function $\phi_0$ defined in this subsection has the same zeros as the function $\phi_0$ defined in (4.370). Thus the zeros of $\phi_0$ are the following

$$0\ , \hat{\mu}_k^{\pm} = \pm(4k-9)\frac{\pi}{4a} + o(1),\ \hat{\mu}_{-k}^{\pm} = \pm i(4k-9)\frac{\pi}{4a} + o(1),\ k = 3, 4, \dots. \tag{4.405}$$

The function $\phi_0$ in this subsection is obtained by multiplying the function $\phi_0$ in Subsection 4.4.3 by $\mu^4$ and by replacing the trigonometric functions with their negatives. This leads to the equation (4.128) for $\mu \neq 0$. Whence the zeros of $\phi_0$ are either real or pure imaginary, see Remark 4.26.

We recall that 0 is a zero of multiplicity 8 of the function $\phi_0$ since 0 is a zero of multiplicity 1 for the function $\widetilde{\psi}_1(\mu) = \tan(\mu a) + \tanh(\mu a)$ defined in Subsection 4.5.1 and it is a zero of multiplicity 7 for the function $\mu \mapsto \psi_0(\mu) = \mu^7$. Whence the zeros of $\phi_0$ counted with multiplicity are the following

$$\left.\begin{array}{l} \hat{\mu}_{-2}^{\pm} = 0,\ \hat{\mu}_{-1}^{\pm} = 0,\ \hat{\mu}_1^{\pm} = 0,\ \hat{\mu}_2^{\pm} = 0, \\ \hat{\mu}_k^{\pm} = \pm(4k-9)\frac{\pi}{4a} + o(1), \\ \hat{\mu}_{-k}^{\pm} = \pm i(4k-9)\frac{\pi}{4a} + o(1),\ k = 3, 4, \dots \end{array}\right\}. \tag{4.406}$$

Let

$$\phi_{00}(\mu) = \cos(\mu a) + \sin(\mu a), \tag{4.407}$$

$$\begin{aligned} \phi_{01}(\mu) = {} & (-1 + \tanh(\mu a))\cos(\mu a) + \frac{i}{\alpha\mu^5}\left(-(1+\mu^4)\cos(\mu a) + \frac{\mu^4 - 1}{\cosh(\mu a)}\right) \\ & + \frac{1}{\alpha^2\mu^6}(\sin(\mu a) - \cos(\mu a)\tanh(\mu a)). \end{aligned} \tag{4.408}$$

Then

$$\phi_{02}(\mu) = \frac{\phi(\mu)}{\alpha^2\mu^7\cosh(\mu a)} = \phi_{00}(\mu) + \phi_{01}(\mu). \tag{4.409}$$

The numbers

$$\mu_k^{00} = \left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right),\ k \in \mathbb{Z} \text{ are the zeros of } \phi_{00},$$

see (4.232) and

$$\widetilde{\mu}_k^{00} = i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right),\ k \in \mathbb{Z} \text{ are the images of } \mu_k^{00} \text{ by the rotation of angle } \frac{\pi}{2},$$

see (4.233).

Let $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ be the rectangles defined in (4.234). We have $\mu_k^{00} = \left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in R_k$, $-\mu_k^{00} = -\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in R_{-k}$, $\widetilde{\mu}_k^{00} = i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_k$ and $-\widetilde{\mu}_k^{00} = -i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_{-k}$, see Remark 4.34. The rectangles $R_k$ do not intersect, as well as the rectangles $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{Z}$, due to $\varepsilon < \frac{\pi}{2a}$. We recall that there exists a constant $\rho(\varepsilon) > 0$ such that $|\phi_{00}(\mu)| > \rho(\varepsilon)$ for all $\mu$ on the rectangle $R_k$, $k \in \mathbb{Z}$, as $|\phi_{00}|$ is periodic of period $\frac{\pi}{a}$.

It follows from (4.235) and (4.236) that for all $\mu$ on the rectangle $R_k$, where $|k| > k_0(\varepsilon)$ is sufficiently large positive, we have

$$|\phi_{01}(\mu)| < \frac{3+\sqrt{2}}{\alpha|\mu|} + \frac{3+\sqrt{2}}{\alpha|\mu|^5} + \frac{3\sqrt{2}}{|\mu|^6} + 3e^{-|\Re\mu a|}. \tag{4.410}$$

Since the right hand tends to 0 as $|\Re\mu a| \to \infty$, then for all $\mu$ on the rectangle $R_k$, where $k \in \mathbb{Z}$, $|k| > k_0(\varepsilon)$,

$$|\phi_{01}(\mu)| < |\phi_{00}(\mu)|. \tag{4.411}$$

For $\mu \in \mathbb{C}$, we have

$$\begin{aligned}
\phi_0(-\mu) &= (-\mu)^7(\sin(-\mu a)\cosh(-\mu a) + \cos(-\mu a)\sinh(-\mu a)) \\
&= -\mu^7(-\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \\
&= \mu^7(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) = \phi_0(\mu)
\end{aligned} \tag{4.412}$$

and

$$\begin{aligned}
\phi_1(-\mu) &= -\mu(\sin(-\mu a)\cosh(-\mu a) - \cos(-\mu a)\sinh(-\mu a)) \\
&- i\alpha((-\mu)^6 + (-\mu)^2)\cos(-\mu a)\cosh(-\mu a) + i\alpha((-\mu)^6 - (-\mu)^2) \\
&= -\mu(-\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \\
&- i\alpha(\mu^6 + \mu^2)\cos(\mu a)\cosh(\mu a) + i\alpha(\mu^6 - \mu^2) \\
&= \mu(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \\
&- i\alpha(\mu^6 + \mu^2)\cos(\mu a)\cosh(\mu a) + i\alpha(\mu^6 - \mu^2) = \phi_1(\mu).
\end{aligned} \tag{4.413}$$

It follows from (4.402), (4.412) and (4.413) that $\phi$ is an even function. Thus the estimates (4.410) and (4.411) follow for all $\mu$ on the rectangle $R_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough. Let

$$\widetilde{\phi}_0(\mu) = \alpha^2\mu^7(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) - i\alpha\mu^6\cos(\mu a)\cosh(\mu a) + i\alpha\mu^6 + \mu(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \text{ and} \tag{4.414}$$

$$\widetilde{\phi}_1(\mu) = -i\alpha\mu^2\cos(\mu a)\cosh(\mu a) - i\alpha\mu^2. \tag{4.415}$$

Then we have

$$\phi(\mu) = \widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu). \tag{4.416}$$

However

$$\begin{aligned}\widetilde{\phi}_0(i\mu) &= \alpha^2(i\mu)^7(\sin(i\mu a)\cosh(i\mu a) + \cos(i\mu a)\sinh(i\mu a)) - i\alpha(i\mu)^6\cos(i\mu a)\cosh(i\mu a)\\ &\quad + i\alpha(i\mu)^6 + i\mu(\sin(i\mu a)\cosh(i\mu a) - \cos(i\mu a)\sinh(i\mu a))\\ &= -i\alpha^2\mu^7(i\sinh(\mu a)\cos(\mu a) + i\cosh(\mu a)\sin(\mu a)) - i\alpha\mu^6\cos(\mu a)\cosh(\mu a)\\ &\quad + i\alpha\mu^6 + i\mu(i\sinh(\mu a)\cos(\mu a) - i\cosh(\mu a)\sin(\mu a))\\ &= \alpha^2\mu^7(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) - i\alpha\mu^6\cos(\mu a)\cosh(\mu a)\\ &\quad + i\alpha\mu^6 + \mu(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) = \widetilde{\phi}_0(\mu),\end{aligned} \tag{4.417}$$

while

$$\begin{aligned}\widetilde{\phi}_1(i\mu) &= -i\alpha(i\mu)^2\cos(i\mu a)\cosh(i\mu a) - i\alpha(i\mu)^2\\ &= +i\alpha\mu^2\cos(\mu a)\cosh(\mu a) + i\alpha\mu^2 = -\widetilde{\phi}_1(\mu).\end{aligned} \tag{4.418}$$

Putting (4.417) and (4.418) together, we have

$$\phi(i\mu) = \widetilde{\phi}_0(\mu) - \widetilde{\phi}_1(\mu). \tag{4.419}$$

It follows from (4.402), (4.403) and (4.404) that we can obtain the estimates (4.410) and (4.411) for all $\mu$ on the rectangles $R_k$ and $R_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$. As $|\phi(\mu)|$ and $|\phi(i\mu)|$ have the same upper-bound $|\widetilde{\phi}_0(\mu)| + |\widetilde{\phi}_1(\mu)|$, see (4.416) and (4.419), then we can obtain the estimates (4.410) and (4.411) for all $\mu$ on the rectangles $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough.

Since the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ are closed curves, 0 is a zero of multiplicity 8 of $\phi_0$, $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$ are simple its zeros, and $\phi_0$ has no nonzero zero in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, then it follows from (4.411), Remark 4.34 and Rouché's theorem that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$, where the asymptotics of the zeros of $\phi_0$ are

$$\left.\begin{aligned} \hat{\mu}_k^{\pm} &= \pm(4k-9)\tfrac{\pi}{4a} + o(1), \text{ where } k \in \mathbb{Z},\ k \geq k_0(\varepsilon) \text{ and} \\ \hat{\mu}_k^{\pm} &= \pm i(4|k|-9)\tfrac{\pi}{4a} + o(1), \text{ where } k \in \mathbb{Z},\ k \leq -k_0(\varepsilon) \end{aligned}\right\}, \tag{4.420}$$

with $o(1) \to 0$ as $|k| \to \infty$, see (4.406).

Let $S_k$, $k \in \mathbb{N}$ be the square defined in (4.66) and

$$\phi_{10}(\mu) = \mu(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)), \tag{4.421}$$

$$\phi_{11}(\mu) = -i\alpha(\mu^6 + \mu^2)\cos(\mu a)\cosh(\mu a), \tag{4.422}$$

$$\phi_{12}(\mu) = i\alpha(\mu^6 - \mu^2). \tag{4.423}$$

Then it follows from (4.253) that there exists $M_0 > 0$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \check{k}_1 = \frac{a}{\pi}\sqrt[6]{\frac{7M_0}{\alpha^2}}$

$$\begin{aligned} |\phi_{10}(\mu)| &= |\mu||\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)| \\ &= \frac{3}{\alpha^2}\frac{|\tan(\mu a) - \tanh(\mu a)|}{|\mu|^6}\frac{\alpha^2}{3}|\mu|^7|\cos(\mu a)\cosh(\mu a)| \\ &\leq \frac{6}{2\alpha^2}\frac{M_0}{|\mu|^6}\frac{\alpha^2}{3}|\mu|^7|\cos(\mu a)\cosh(\mu a)| \\ &\leq \frac{6}{\alpha^2}\frac{M_0}{|\mu|^6}\frac{\alpha^2}{3}|\mu|^7|\cos(\mu a)\cosh(\mu a)||\tan(\mu a) + \tanh(\mu a)| < \frac{\alpha^2}{3}|\phi_0(\mu)|, \end{aligned} \tag{4.424}$$

while there exists $\widetilde{k}_1 = \frac{13a}{\alpha\pi}$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \widetilde{k}_1$,

$$\begin{aligned} |\phi_{11}(\mu)| &= \alpha|\mu^6 + \mu^2||\cos(\mu a)\cosh(\mu a)| \\ &\leq 2\alpha|\mu|^6|\cos(\mu a)\cosh(\mu a)| \\ &\leq \frac{12}{\alpha|\mu|}\frac{\alpha^2}{3}|\mu|^7||\cos(\mu a)\cosh(\mu a)||\tan(\mu a) + \tanh(\mu a)| < \frac{\alpha^2}{3}|\phi_0(\mu)| \end{aligned} \tag{4.425}$$

and finally for all $\mu$ on the rectangle $S_k$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_1 = \max\{\widetilde{k}_1, \frac{a}{\pi}\frac{13}{\alpha}\}$

$$\begin{aligned}
|\phi_{12}(\mu)| &= \alpha|\mu^6 - \mu^2| \\
&< 2\alpha|\mu|^6 \\
&\leq \frac{2}{3}\alpha|\mu|^6|\cos(\mu a)\cosh(\mu a)| \\
&\leq \frac{1}{\alpha|\mu|}\frac{1}{3}\alpha|\mu|^7||\cos(\mu a)\cosh(\mu a)||\tan(\mu a) + \tanh(\mu a)| \\
&\leq \frac{\alpha^2}{3}|\phi_0(\mu)|.
\end{aligned} \tag{4.426}$$

Putting together (4.424), (4.425) and (4.426), we have for all $\mu$ on the square $S_k$, where $k \geq \hat{k}_1 = \max\{\check{k}_1, \widetilde{k}_1, \widetilde{m}_1\}$

$$|\phi_1(\mu)| \leq |\phi_{10}(\mu)| + |\phi_{11}(\mu)| + |\phi_{12}(\mu)| < \alpha^2|\phi_0(\mu)|. \tag{4.427}$$

It follows from (4.427) and Rouché's theorem that $\phi_0$ and $\phi$ have the same number of zeros inside the square $S_k$, where $k \geq \hat{k}_1$.

**Proposition 4.48.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y''(0)$, $B_2(y) = y^{(3)}(0)$, $B_3y = y''(a) + i\alpha\lambda y'(a)$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{cases}
\hat{\mu}_k^{\pm} & = \pm(4k-9)\frac{\pi}{4a} + o(1), \quad if\ k > 0, \\
\hat{\mu}_k^{\pm} & = \pm i(4|k|-9)\frac{\pi}{4a} + o(1), \quad if\ k < 0.
\end{cases}$$

*In particular, there is an even number of pure imaginary eigenvalues.*

**Remark 4.49.** We give the enumeration of the zeros of $\phi$ inside the square $S_{k_1}$, where $k_1 = \max\{\hat{k}_1, k_0(\varepsilon)\}$. The remainder of the proof is identical to the remainder of the proof derived for Proposition 4.23.

*Proof.* The number of zeros of $\phi_0$ inside the square $S_{k_1}$ is $4k_1$, as 0 is a zero of multiplicity 8, while $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$ are simple zeros. Thus the number of zeros of $\phi$ inside the square $S_{k_1}$ is

$4k_1$ and these zeros are the following

$$\hat{\mu}_1^{\pm} = \pm\left(-\frac{5\pi}{4a}\right) + o(1),\ \hat{\mu}_2^{\pm} = \pm\left(-\frac{\pi}{4a}\right) + o(1), \ldots, \hat{\mu}_{k_1}^{\pm} = \pm(4k_1 - 9)\frac{\pi}{4a} + o(1),$$
$$\hat{\mu}_{-1}^{\pm} = \pm i\left(-\frac{5\pi}{4a}\right) + o(1),\ \hat{\mu}_{-2}^{\pm} = \pm i\left(-\frac{\pi}{4a}\right) + o(1), \ldots, \hat{\mu}_{k_1}^{\pm} = \pm(4k_1 - 9)\frac{\pi}{4a} + o(1).$$

Thus

$$\hat{\mu}_k^{\pm} = \pm(4k - 9)\frac{\pi}{4a} + o(1),\ \text{where } 0 \le k \le k_1,$$
$$\hat{\mu}_k^{\pm} = \pm i(4|k| - 9)\frac{\pi}{4a} + o(1),\ \text{where } -k_1 \le k \le -1.$$

Using the approach of the proof of Proposition 4.23 we can prove that the zeros of $\phi$ for $|k| \ge k_1$ are

$$\hat{\mu}_k^{\pm},\ k = k_1, k_1 + 1, \ldots$$
$$\hat{\mu}_k^{\pm},\ k = -k_1, -k_1 - 1, \ldots$$

and satisfy

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(4k - 9)\frac{\pi}{4a} + o(1), \text{ where } k \ge k_1, \\ \hat{\mu}_k^{\pm} = \pm i(4|k| - 9)\frac{\pi}{4a} + o(1), \text{ where } k \le -k_1, \end{cases}$$

see (4.420). □

### 4.6.3 Asymptotic of the eigenvalues for $B_3 y = y'(a) - i\alpha\lambda y''(a)$ and $B_4 y = y^{(3)}(a) - i\alpha\lambda y(a)$

It follows from (4.359) that

$$\begin{aligned}
\det M &= B_3 y_1 B_4 y_2 - B_4 y_1 B_3 y_2 \\
&= (y_1'(a) - i\alpha\mu^2 y_1''(a))(y_2^{(3)}(a) - i\alpha\mu^2 y_2(a)) - (y_1^{(3)}(a) - i\alpha\mu^2 y_1(a))(y_2'(a) - i\alpha\mu^2 y_2''(a)) \\
&= y_1'(a)y_2^{(3)}(a) - \alpha^2\mu^4 y_1''(a)y_2(a) - y_1^{(3)}(a)y_2'(a) + \alpha^2\mu^4 y_1(a)y_2''(a) - i\alpha\mu^2(y_1''(a)y_2^{(3)}(a) \\
&+ y_1'(a)y_2(a) - y_1^{(3)}(a)y_2''(a) - y_1(a)y_2'(a)) \\
&= y_1'(a)y_2^{(3)}(a) - y_1^{(3)}(a)y_2'(a) - \alpha^2\mu^4 y_1''(a)y_2(a) + \alpha^2\mu^4 y_1(a)y_2''(a) \\
&- i\alpha\mu^2(y_1''(a)y_2^{(3)}(a) + y_1'(a)y_2(a) - y_1(a)y_2'(a) - y_2''(a)y_1^{(3)}(a)).
\end{aligned} \tag{4.428}$$

It follows from (4.361) and (4.362) that

$$\begin{aligned}\det M &= \mu^8 y_4(a)y_4'(a) - \mu^4 y_4''(a)y_4^{(3)}(a) - \alpha^2\mu^8 y_4''(a)y_4'(a) + \alpha^2\mu^8 y_4^{(3)}(a)y_4(a)\\ &\quad - i\alpha\mu^2(\mu^8(y_4'(a))^2 + \mu^4 y_4(a)y_4''(a) - (y_4^{(3)}(a))^2 - \mu^8 y_4(a)y_4''(a)). \end{aligned} \tag{4.429}$$

Let

$$A_1(a) = \mu^8 y_4(a)y_4'(a) - \mu^4 y_4''(a)y_4^{(3)}(a) - \alpha^2\mu^8 y_4''(a)y_4'(a) + \alpha^2\mu^8 y_4^{(3)}(a)y_4(a) \text{ and} \tag{4.430}$$

$$A_2(a) = \mu^8(y_4'(a))^2 + \mu^4 y_4(a)y_4''(a) - (y_4^{(3)}(a))^2 - \mu^8 y_4(a)y_4''(a). \tag{4.431}$$

Then it follows from (4.28), (4.29), (4.212) and (4.213) that

$$\begin{aligned}A_1(a) &= \mu^8\left(\frac{1}{4\mu^5}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^5}\sin(\mu a)\cosh(\mu a) - \frac{1}{4\mu^5}\sinh(\mu a)\cos(\mu a)\right.\\ &\quad \left.+\frac{1}{4\mu^5}\sinh(\mu a)\cosh(\mu a)\right) - \mu^4\left(\frac{1}{4\mu}\sin(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sinh(\mu a)\cos(\mu a)\right.\\ &\quad \left.+\frac{1}{4\mu}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu}\sinh(\mu a)\cosh(\mu a)\right) - \alpha^2\mu^8\left(-\frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a)\right.\\ &\quad \left.-\frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a)\right)\\ &\quad +\alpha^2\mu^8\left(-\frac{1}{4\mu^3}\sin(\mu a)\cos(\mu a) + \frac{1}{4\mu^3}\sinh(\mu a)\cos(\mu a) - \frac{1}{4\mu^3}\sin(\mu a)\cosh(\mu a)\right.\\ &\quad \left.+\frac{1}{4\mu^3}\sinh(\mu a)\cosh(\mu a)\right)\\ &= -\frac{\mu^3}{2}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) - \frac{\alpha^2\mu^5}{2}(\sin(\mu a)\cosh(\mu a)\\ &\quad - \cos(\mu a)\sinh(\mu a)),\end{aligned} \tag{4.432}$$

while (4.31), (4.33) and (4.211) give

$$\begin{aligned}A_2(a) &= \mu^8\left(\frac{1}{4\mu^4}\cos^2(\mu a) - \frac{1}{2\mu^4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4\mu^4}\cosh^2(\mu a)\right)\\ &\quad - \left(\frac{1}{4}\cos^2(\mu a) + \frac{2}{4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4}\cosh^2(\mu a)\right)\\ &\quad + \mu^4\left(-\frac{1}{4\mu^4}\sin^2(\mu a) + \frac{1}{4\mu^4}\sinh^2(\mu a)\right) - \mu^8\left(-\frac{1}{4\mu^4}\sin^2(\mu a)\right.\\ &\quad \left.+\frac{1}{4\mu^4}\sinh^2(\mu a)\right)\\ &= \frac{\mu^4-1}{2} - \frac{\mu^4+1}{2}\cos(\mu a)\cosh(\mu a).\end{aligned} \tag{4.433}$$

It follows from (4.429)–(4.433) that

$$\begin{aligned}\det M = &-\frac{i\alpha\mu^6}{2}\cos(\mu a)\cosh(\mu a) - \frac{\mu^3}{2}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a))\\ &- \frac{\alpha^2\mu^5}{2}(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) + \frac{i\alpha\mu^6}{2} - \frac{i\alpha\mu^2}{2}(1 + \cos(\mu a)\cosh(\mu a)).\end{aligned} \tag{4.434}$$

Thus the characteristic equation $2i \det M = 0$ is

$$\phi(\mu) := \alpha\phi_0(\mu) + \phi_1(\mu) = 0, \tag{4.435}$$

where

$$\phi_0(\mu) = \mu^6 \cos(\mu a)\cosh(\mu a), \tag{4.436}$$

$$\begin{aligned}\phi_1(\mu) = &-\alpha\mu^6 - i\alpha^2\mu^5(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a))\\ &- i\mu^3(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) + \alpha\mu^2(1 + \cos(\mu a)\cosh(\mu a)).\end{aligned} \tag{4.437}$$

The function $\phi_0$ defined in this subsection has the same zeros as the function $\phi_0$ defined in (4.89). However 0 is a zero of multiplicity 6 in this section, while it is a zero of multiplicity 4 for the function $\phi_0$ defined in (4.89). Thus the zeros of $\phi_0$ counted with multiplicity are

$$\left.\begin{array}{l}\mu^{\pm}_{-1} = 0,\ \mu^{\pm}_0 = 0,\ \mu^{\pm}_1 = 0,\ \mu^{\pm}_k = \pm(2k-3)\frac{\pi}{2a}\\ \text{and } \mu^{\pm}_{-k} = \pm i(2k-3)\frac{\pi}{2a},\ \text{where } k = 2, 3, \ldots\end{array}\right\}, \tag{4.438}$$

see (4.91).

Let $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ be the rectangles defined in (4.47) and

$$\phi_{10}(\mu) = -i\mu^3(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)), \tag{4.439}$$

$$\phi_{11}(\mu) = -i\alpha^2\mu^5(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)), \tag{4.440}$$

$$\phi_{12}(\mu) = \alpha\mu^2\cos(\mu a)\cosh(\mu a), \tag{4.441}$$

$$\phi_{13}(\mu) = -\alpha(\mu^6 - \mu^2). \tag{4.442}$$

It follows from (4.48) and (4.50) that there exist $\beta_1 > 0$, $\beta_2 \geq 1$ and $j_0 \in \mathbb{N}$, such that for all

$\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq m_0 = \max\left\{j_0, \frac{a}{\pi}\sqrt[3]{\frac{5(\beta_1+\beta_2)}{\alpha}}\right\}$,

$$\begin{aligned}|\phi_{10}(\mu)| &= |\mu|^3|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\ &\leq \frac{4(\beta_1+\beta_2)}{\alpha|\mu|^3}\frac{\alpha}{4}|\mu|^6|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{4(\beta_1+\beta_2)}{\alpha|\mu|^3}\frac{\alpha}{4}|\phi_0(\mu)| < \frac{\alpha}{4}|\phi_0(\mu)|, \end{aligned} \tag{4.443}$$

while for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \bar{m}_0 = \max\left\{j_0, \frac{5a\alpha(\beta_1+\beta_2)}{\pi}\right\}$,

$$\begin{aligned}|\phi_{11}(\mu)| &= \alpha^2|\mu|^5|\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)| \\ &\leq \frac{4\alpha(\beta_1+\beta_2)}{|\mu|}\frac{\alpha}{4}|\mu|^6|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{4\alpha(\beta_1+\beta_2)}{|\mu|}\frac{\alpha}{4}|\phi_0(\mu)| < \frac{\alpha}{4}|\phi_0(\mu)|, \end{aligned} \tag{4.444}$$

and for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \breve{m}_0 = \max\left\{j_0, \frac{a}{\pi}\sqrt[4]{5}\right\}$, we have

$$\begin{aligned}|\phi_{12}(\mu)| &= \alpha|\mu|^2|\cos(\mu a)\cosh(\mu a)| \\ &\leq \frac{4}{|\mu|^4}\frac{\alpha}{4}|\mu|^6|\cos(\mu a)\cosh(\mu a)| < \frac{\alpha}{4}|\phi_0(\mu)| \end{aligned} \tag{4.445}$$

finally for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_0 = \max\{j_0, \frac{a}{\pi}\ln 14\}$

$$\begin{aligned}|\phi_{13}(\mu)| &= \alpha|\mu^6 - \mu^2| \\ &< 2\alpha|\mu|^6 \\ &< \frac{1}{|\mu|}\frac{\alpha}{4}|\mu|^7|\cos(\mu a)\cosh(\mu a)| < \frac{\alpha}{4}|\phi_0(\mu)|. \end{aligned} \tag{4.446}$$

Putting together (4.443), (4.444), (4.445) and (4.446), we have for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_0 = \max\{m_0, \bar{m}_0, \breve{m}_0, \widetilde{m}_0\}$

$$|\phi_1(\mu)| \leq |\phi_{10}(\mu)| + |\phi_{11}(\mu)| + |\phi_{12}(\mu)| + |\phi_{13}(\mu)| < \alpha|\phi_0(\mu)|. \tag{4.447}$$

Since the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ are closed curves, 0 is a zero of multiplicity 6 of $\phi_0$, while $\mu_k^{\pm}$ and $\mu_{-k}^{\pm}$, where $k = 2, 3, \ldots$ are simple zeros, then it follows from (4.447) and

Rouché's theorem that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$ where the asymptotics of the zeros of $\phi_0$

$$\left.\begin{array}{l} \hat{\mu}_k^{\pm} = \pm(2k-3)\frac{\pi}{2a} + o(1), \text{ where } k \in \mathbb{Z}, \text{ and } k \geq \hat{m}_0 \\ \hat{\mu}_k^{\pm} = \pm i(2|k|-3)\frac{\pi}{2a} + o(1), \text{ where } k \in \mathbb{Z}, \text{ and } k \leq -\hat{m}_0 \end{array}\right\}, \tag{4.448}$$

with $o(1) \to 0$ as $|k| \to \infty$, see (4.438).

Let $S_k$, $k \in \mathbb{N}$ be the square defined in (4.66). Then it follows from (4.68) and (4.69) that there exist $j_1 \in \mathbb{N}$, $\delta_1 > 0$ and $\delta_2 > 1$ such that, for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq m_1 = \max\left\{j_1, \frac{\pi}{a}\sqrt[3]{\frac{5(\delta_1+\delta_2)}{\alpha}}\right\}$

$$\begin{aligned} |\phi_{10}(\mu)| &= |\mu|^3|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\ &\leq \frac{4}{\alpha|\mu|^3}(\delta_1+\delta_2)\frac{\alpha}{4}|\mu|^6|\cos(\mu a)\cosh(\mu a)| < \frac{\alpha}{4}|\phi_0(\mu)|, \end{aligned} \tag{4.449}$$

while for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \bar{m}_1 = \max\left\{j_1, \frac{5a\alpha}{\pi}(\delta_1+\delta_2)\right\}$

$$\begin{aligned} |\phi_{11}(\mu)| &= \alpha^2|\mu|^5|\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)| \\ &\leq \frac{4\alpha(\delta_1+\delta_2)}{|\mu|}\frac{\alpha}{4}|\mu|^6|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{4\alpha(\delta_1+\delta_2)}{|\mu|}\frac{\alpha}{4}|\phi_0(\mu)| < \frac{\alpha}{4}|\phi_0(\mu)|, \end{aligned} \tag{4.450}$$

and for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \breve{m}_1 = \max\left\{j_1, \frac{a}{\pi}\sqrt[4]{5}\right\}$, we have

$$\begin{aligned} |\phi_{12}(\mu)| &= \alpha|\mu|^2|\cos(\mu a)\cosh(\mu a)| \\ &\leq \frac{4}{|\mu|^4}\frac{\alpha}{4}|\mu|^6|\cos(\mu a)\cosh(\mu a)| < \frac{\alpha}{4}|\phi_0(\mu)|, \end{aligned} \tag{4.451}$$

finally for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_1 = \max\{j_1, \frac{a}{\pi}\ln 14\}$

$$\begin{aligned} |\phi_{13}(\mu)| &= \alpha|\mu^6 - \mu^2| \\ &< 2\alpha|\mu|^6 \\ &< \frac{1}{|\mu|}\frac{\alpha}{4}|\mu|^7|\cos(\mu a)\cosh(\mu a)| < \frac{\alpha}{4}|\phi_0(\mu)|. \end{aligned} \tag{4.452}$$

Putting together (4.449), (4.450), (4.451) and (4.452), we have for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_1 = \max\{m_1, \bar{m}_1, \breve{m}_1, \widetilde{m}_1\}$

$$|\phi_1(\mu)| \leq |\phi_{10}(\mu)| + |\phi_{11}(\mu)| + |\phi_{12}(\mu)| + |\phi_{13}(\mu)| < \alpha|\phi_0(\mu)|. \tag{4.453}$$

As the square $S_k$, $k \in \mathbb{N}$ is a closed curve, then it follows from (4.453) and Rouché's theorem that the functions $\phi_0$ and $\phi$ have the same number of zeros inside the square $S_k$.

**Remark 4.50.** Let $k_1 = \max\{\hat{m}_0, \hat{m}_1\}$. As 0 is a zero of multiplicity 6 of $\phi_0$, while $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$, where $k = 2, 3 \ldots$, are its simple zeros, then the number of zeros of $\phi_0$ and therefore of $\phi$ inside the square $S_k$, where $k \in \mathbb{N}$ and $k \geq k_1$ is $4k + 2$. Thus this proposition follows.

**Proposition 4.51.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y''(0)$, $B_2(y) = y^{(3)}(0)$, $B_3 y = y'(a) - i\alpha\lambda y''(a)$ and $B_4 y = y^{(3)}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{cases} \hat{\mu}_k^{\pm} & = \pm(2k-3)\frac{\pi}{2a} + o(1), \quad if\ k > 0, \\ \hat{\mu}_k^{\pm} & = \pm i(2|k|-3)\frac{\pi}{2a} + o(1), \quad if\ k < 0. \end{cases}$$

*In particular, there is an odd number of pure imaginary eigenvalues.*

**Remark 4.52.** We give only the enumeration of the zeros of $\phi$ inside the the square $S_{k_1}$. The remainder of the proof of the above proposition is identical to the remainder of the proof derived for Proposition 4.23.

*Proof.* The function $\phi$ has $4k_1 + 2$ number of zeros inside the square $S_{k_1}$. These zeros are the following

$$\hat{\mu}_0^{\pm} = \pm\left(-\frac{3\pi}{2a}\right) + o(1),\ \hat{\mu}_1^{\pm} = \pm\left(-\frac{\pi}{2a}\right) + o(1), \ldots, \hat{\mu}_{k_1}^{\pm} = \pm(2k_1 - 3)\frac{\pi}{2a} + o(1),$$
$$\hat{\mu}_{-1}^{\pm} = \pm i\left(-\frac{\pi}{2a}\right) + o(1),\ \hat{\mu}_{-2}^{\pm} = \pm i\left(\frac{\pi}{2a}\right) + o(1), \ldots, \hat{\mu}_{-(k_1)}^{\pm} = \pm i(2k_1 - 3)\frac{\pi}{2a} + o(1).$$

Thus

$$\hat{\mu}_k^{\pm} = \pm(2k-3)\frac{\pi}{2a} + o(1), \text{ where } 1 \leq k \leq k_1 + 1,$$
$$\hat{\mu}_{-k}^{\pm} = \pm i(2|k|-3)\frac{\pi}{2a} + o(1), \text{ where } -(k_1+1) \leq k \leq -1.$$

Using the approach of the proof of Proposition 4.23, we can show that the zeros of $\phi$ for

$k \geq k_1$ are

$$\hat{\mu}_k^{\pm},\ k = k_1, k_1+1, \ldots$$
$$-k_1, -k_1-1, \ldots$$

and satisfy

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(2k-3)\frac{\pi}{2a} + o(1), \text{ where } k \geq k_1, \\ \hat{\mu}_{-k}^{\pm} = \pm i(2|k|-3)\frac{\pi}{2a} + o(1), \text{ where } k \leq -k_1, \end{cases}$$

see (4.448). □

### 4.6.4 Asymptotic of the eigenvalues for $B_3y = y'(a) - i\alpha\lambda y''(a)$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a)$

It follows from (4.359) that

$$\begin{aligned}
\det M &= B_3y_1B_4y_2 - B_4y_1B_3y_2 \\
&= (y_1'(a) - i\alpha\mu^2 y_1''(a))(y_2(a) + i\alpha\mu^2 y_2^{(3)}(a)) - (y_1(a) + i\alpha\mu^2 y_1^{(3)}(a))(y_2'(a) - i\alpha\mu^2 y_2''(a)) \\
&= y_1'(a)y_2(a) + \alpha^2\mu^4 y_1''(a)y_2^{(3)}(a) - y_1(a)y_2'(a) - \alpha^2\mu^4 y_1^{(3)}(a)y_2''(a) + i\alpha\mu^2(y_1'(a)y_2^{(3)}(a) \\
&- y_1''(a)y_2(a) + y_1(a)y_2''(a) - y_1^{(3)}(a)y_2'(a)) \\
&= y_1'(a)y_2(a) - y_1(a)y_2'(a) + \alpha^2\mu^4 y_1''(a)y_2^{(3)}(a) - \alpha^2\mu^4 y_1^{(3)}(a)y_2''(a) \\
&+ i\alpha\mu^2(y_1'(a)y_2^{(3)}(a) - y_1''(a)y_2(a) - y_1^{(3)}(a)y_2'(a) + y_2''(a)y_1(a)).
\end{aligned} \tag{4.454}$$

It follows from (4.361) and (4.362) that

$$\begin{aligned}
\det M = \mu^4 y_4(a)y_4''(a) - (y_4^{(3)}(a))^2 + \alpha^2\mu^{12}(y_4'(a))^2 - \alpha^2\mu^{12}y_4''(a)y_4(a) \\
+ i\alpha\mu^2(\mu^8 y_4(a)y_4'(a) - \mu^4 y_4'(a)y_4''(a) - \mu^4 y_4''(a)y_4^{(3)}(a) + \mu^4 y_4(a)y_4^{(3)}(a)).
\end{aligned} \tag{4.455}$$

Let

$$A_1(a) = \mu^4 y_4(a)y_4''(a) - (y_4^{(3)}(a))^2 + \alpha^2\mu^{12}(y_4'(a))^2 - \alpha^2\mu^{12}y_4''(a)y_4(a) \text{ and} \tag{4.456}$$
$$A_2(a) = \mu^8 y_4(a)y_4'(a) - \mu^4 y_4'(a)y_4''(a) - \mu^4 y_4''(a)y_4^{(3)}(a) + \mu^4 y_4(a)y_4^{(3)}(a). \tag{4.457}$$

Then it follows from (4.31), (4.33) and (4.211) that

$$\begin{aligned} A_1(a) &= \mu^4 \left( -\frac{1}{4\mu^4} \sin^2(\mu a) + \frac{1}{4\mu^4} \sinh^2(\mu a) \right) - \alpha^2 \mu^{12} \left( -\frac{1}{4\mu^4} \sin^2(\mu a) \right. \\ &\quad \left. + \frac{1}{4\mu^4} \sinh^2(\mu a) \right) + \alpha^2 \mu^{12} \left( \frac{1}{4\mu^4} \cos^2(\mu a) - \frac{1}{2\mu^4} \cos(\mu a) \cosh(\mu a) \right. \\ &\quad \left. + \frac{1}{4\mu^4} \cosh^2(\mu a) \right) - \left( \frac{1}{4} \cos^2(\mu a) + \frac{2}{4} \cos(\mu a) \cosh(\mu a) \right. \\ &\quad \left. + \frac{1}{4} \cosh^2(\mu a) \right) \\ &= -\frac{\alpha^2 \mu^8}{2} \cos(\mu a) \cosh(\mu a) + \frac{\alpha^2 \mu^8}{2} - \frac{1}{2} \cos(\mu a) \cosh(\mu a) - \frac{1}{2}, \end{aligned} \tag{4.458}$$

while (4.28), (4.29), (4.212) and (4.213) give

$$\begin{aligned} A_2(a) &= -\mu^4 \left( -\frac{1}{4\mu^3} \sin(\mu a) \cos(\mu a) - \frac{1}{4\mu^3} \sinh(\mu a) \cos(\mu a) + \frac{1}{4\mu^3} \sin(\mu a) \cosh(\mu a) \right. \\ &\quad \left. + \frac{1}{4\mu^3} \sinh(\mu a) \cosh(\mu a) \right) + \mu^4 \left( -\frac{1}{4\mu^3} \sin(\mu a) \cos(\mu a) - \frac{1}{4\mu^3} \sin(\mu a) \cosh(\mu a) \right. \\ &\quad \left. + \frac{1}{4\mu^3} \sinh(\mu a) \cos(\mu a) + \frac{1}{4\mu^3} \sinh(\mu a) \cosh(\mu a) \right) + \mu^8 \left( \frac{1}{4\mu^5} \sin(\mu a) \cos(\mu a) \right. \\ &\quad \left. - \frac{1}{4\mu^5} \sin(\mu a) \cosh(\mu a) - \frac{1}{4\mu^5} \sinh(\mu a) \cos(\mu a) + \frac{1}{4\mu^5} \sinh(\mu a) \cosh(\mu a) \right) \\ &\quad - \mu^4 \left( \frac{1}{4\mu} \sin(\mu a) \cos(\mu a) + \frac{1}{4\mu} \sinh(\mu a) \cos(\mu a) + \frac{1}{4\mu} \sin(\mu a) \cosh(\mu a) \right. \\ &\quad \left. + \frac{1}{4\mu} \sinh(\mu a) \cosh(\mu a) \right) \\ &= -\frac{\mu^3}{2} (\sin(\mu a) \cosh(\mu a) + \cos(\mu a) \sinh(\mu a)) \\ &\quad - \frac{\mu}{2} (\sin(\mu a) \cosh(\mu a) - \cos(\mu a) \sinh(\mu a)). \end{aligned} \tag{4.459}$$

Putting together (4.455), (4.456), (4.457), (4.458) and (4.459) we have

$$\begin{aligned} \det M &= -\frac{\alpha^2 \mu^8}{2} \cos(\mu a) \cosh(\mu a) + \frac{\alpha^2 \mu^8}{2} - \frac{i\alpha\mu^5}{2} (\sin(\mu a) \cosh(\mu a) + \cos(\mu a) \sinh(\mu a)) \\ &\quad - \frac{i\alpha\mu^3}{2} (\sin(\mu a) \cosh(\mu a) - \cos(\mu a) \sinh(\mu a)) - \frac{1}{2} \cos(\mu a) \cosh(\mu a) - \frac{1}{2}. \end{aligned} \tag{4.460}$$

Thus the characteristic equation $-2 \det M = 0$ is

$$\phi(\mu) := \alpha^2 \phi_0(\mu) + \phi_1(\mu) = 0, \tag{4.461}$$

where

$$\phi_0(\mu) = \mu^8 \cos(\mu a)\cosh(\mu a) \tag{4.462}$$

$$\phi_1(\mu) = -\alpha^2\mu^8 + i\alpha\mu^5(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \tag{4.463}$$

$$+ i\alpha\mu^3(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) + \cos(\mu a)\cosh(\mu a) + 1. \tag{4.464}$$

The function $\phi_0$ defined in this subsection has the same zeros as the function $\phi_0$ defined in (4.89). However 0 is a zero of multiplicity 8 in this section, while it is a zero of multiplicity 4 for the function $\phi_0$ defined in (4.89). Thus the zeros of $\phi_0$ counted with multiplicity are

$$\left.\begin{array}{l} \mu_{-2}^{\pm} = 0,\ \mu_{-1}^{\pm} = 0,\ \mu_1^{\pm} = 0,\ \ \mu_2^{\pm} = 0,\ \mu_k^{\pm} = \pm(2k-5)\frac{\pi}{2a} \\ \text{and } \mu_{-k}^{\pm} = \pm(2k-5)\frac{\pi}{2a},\ \text{where } k = 3, 4, \ldots \end{array}\right\}, \tag{4.465}$$

see (4.91).

Let

$$\phi_{10}(\mu) = i\alpha\mu^5(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)), \tag{4.466}$$

$$\phi_{11}(\mu) = i\alpha\mu^3(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)), \tag{4.467}$$

$$\phi_{12}(\mu) = \cos(\mu a)\cosh(\mu a), \tag{4.468}$$

$$\phi_{13}(\mu) = -\alpha^2\mu^8 + 1. \tag{4.469}$$

Let $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ be the rectangles defined in (4.47). It follows from (4.48) and (4.50) that there exist $\beta_1 > 0$, $\beta_2 \geq 1$ and $j_0 > 0$, such that for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq m_0 = \max\left\{j_0, \frac{a}{\pi}\sqrt[3]{\frac{5(\beta_1+\beta_2)}{\alpha}}\right\}$,

$$\begin{aligned} |\phi_{10}(\mu)| &= 4\frac{1}{4}\alpha|\mu|^5|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\ &\leq \frac{4(\beta_1+\beta_2)}{\alpha|\mu|^3}\frac{\alpha^2}{4}|\mu|^8|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{4(\beta_1+\beta_2)}{\alpha|\mu|^3}\frac{\alpha^2}{4}|\phi_0(\mu)| < \frac{\alpha^2}{4}|\phi_0(\mu)|, \end{aligned} \tag{4.470}$$

while for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \bar{m}_0 =$

$\max\left\{j_0, \frac{a}{\pi}\sqrt[5]{\frac{5(\beta_1+\beta_2)}{\alpha}}\right\}$,

$$\begin{aligned}|\phi_{11}(\mu)| &= \alpha|\mu|^3|\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)| \\ &\le \frac{4(\beta_1+\beta_2)}{\alpha|\mu|^5}\frac{\alpha^2}{4}|\mu|^8|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{4(\beta_1+\beta_2)}{\alpha|\mu|^5}\frac{\alpha^2}{4}|\phi_0(\mu)| < \frac{\alpha^2}{4}|\phi_0(\mu)|, \end{aligned} \tag{4.471}$$

and for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \ge \check{m}_0 = \max\left\{j_0, \frac{a}{\pi}\sqrt[8]{\frac{5}{\alpha^2}}\right\}$, we have

$$\begin{aligned}|\phi_{12}(\mu)| &= |\cos(\mu a)\cosh(\mu a)| \\ &\le \frac{4}{\alpha^2|\mu|^8}\frac{\alpha^2}{4}|\mu|^8|\cos(\mu a)\cosh(\mu a)| < \frac{\alpha^2}{4}|\phi_0(\mu)|, \end{aligned} \tag{4.472}$$

finally for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \ge \widetilde{m}_0 = \max\{j_0, \frac{a}{\pi}\ln 14\}$

$$\begin{aligned}|\phi_{13}(\mu)| &< |\alpha^2\mu^8 + 1| \le 2\alpha^2|\mu|^8 \\ &\le \frac{\alpha^2}{4}|\mu|^8|\cos(\mu a)\cosh(\mu a)| < \frac{\alpha^2}{4}|\phi_0(\mu)|. \end{aligned} \tag{4.473}$$

Putting together (4.470), (4.471), (4.472) and (4.473), we have for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \ge \hat{m}_0 = \max\{m_0, \bar{m}_0, \check{m}_0, \widetilde{m}_0\}$

$$|\phi_1(\mu)| \le |\phi_{10}(\mu)| + |\phi_{11}(\mu)| + |\phi_{12}(\mu)| + |\phi_{13}(\mu)| < \alpha^2|\phi_0(\mu)|. \tag{4.474}$$

Since the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ are closed curves, 0 is a zero of multiplicity 8 of $\phi_0$, while $\mu_k^{\pm}$ and $\mu_{-k}^{\pm}$, where $k = 2, 3, \dots$ are simple zeros, then it follows from (4.474) and Rouché's theorem that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$ where the asymptotics of the zeros of $\phi_0$

$$\left.\begin{aligned}\hat{\mu}_k^{\pm} &= \pm(2k-5)\tfrac{\pi}{2a} + o(1), \text{ where } k \in \mathbb{Z}, \text{ and } k \ge \hat{m}_0 \\ \hat{\mu}_k^{\pm} &= \pm i(2|k|-5)\tfrac{\pi}{2a} + o(1), \text{ where } k \in \mathbb{Z}, \text{ and } k \le -\hat{m}_0\end{aligned}\right\}, \tag{4.475}$$

with $o(1) \to 0$ as $|k| \to \infty$, see (4.465).

Let $S_k$, $k \in \mathbb{N}$ be the square defined in (4.66). Then it follows from (4.68) and (4.69) that there exist $j_1 > 0$, $\delta_1 > 0$ and $\delta_2 > 1$ such that, for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and

$k \geq m_1 = \max\left\{j_1, \frac{\pi}{a}\sqrt[3]{\frac{5(\delta_1+\delta_2)}{\alpha}}\right\}$

$$\begin{aligned}|\phi_{10}(\mu)| &= 4\frac{1}{4}\alpha|\mu|^5|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\ &\leq \frac{4(\delta_1+\delta_2)}{\alpha|\mu|^3}\frac{\alpha^2}{4}|\mu|^8|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{4(\delta_1+\delta_2)}{\alpha|\mu|^3}\frac{\alpha^2}{4}|\phi_0(\mu)| < \frac{\alpha^2}{4}|\phi_0(\mu)|, \end{aligned} \tag{4.476}$$

while for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \bar{m}_1 = \max\left\{j_1, \frac{\pi}{a}\sqrt[5]{\frac{5(\delta_1+\delta_2)}{\alpha}}\right\}$

$$\begin{aligned}|\phi_{11}(\mu)| &= \alpha|\mu|^3|\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)| \\ &\leq \frac{4(\delta_1+\delta_2)}{\alpha|\mu|^5}\frac{\alpha^2}{4}|\mu|^8|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{4(\delta_1+\delta_2)}{\alpha|\mu|^5}\frac{\alpha^2}{4}|\phi_0(\mu)| < \frac{\alpha^2}{4}|\phi_0(\mu)|, \end{aligned} \tag{4.477}$$

and for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \breve{m}_1 = \max\left\{j_1, \frac{a}{\pi}\sqrt[8]{\frac{5}{\alpha^2}}\right\}$, we have

$$\begin{aligned}|\phi_{12}(\mu)| &= |\cos(\mu a)\cosh(\mu a)| \\ &\leq \frac{4}{\alpha^2|\mu|^8}\frac{\alpha^2}{4}|\mu|^8|\cos(\mu a)\cosh(\mu a)| < \frac{\alpha^2}{4}|\phi_0(\mu)|, \end{aligned} \tag{4.478}$$

finally for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_1 = \max\{j_1, \frac{a}{\pi}\ln 14\}$

$$\begin{aligned}|\phi_{13}(\mu)| &< |\alpha^2\mu^8 + 1| \leq 2\alpha^2|\mu|^8 \\ &< \frac{\alpha^2}{4}|\mu|^8|\cos(\mu a)\cosh(\mu a)| < \frac{\alpha^2}{4}|\phi_0(\mu)|. \end{aligned} \tag{4.479}$$

Putting together (4.476), (4.477), (4.478) and (4.479), we have for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_1 = \max\{m_1, \bar{m}_1, \breve{m}_1, \widetilde{m}_1\}$

$$|\phi_1(\mu)| \leq |\phi_{10}(\mu)| + |\phi_{11}(\mu)| + |\phi_{12}(\mu)| + |\phi_{13}(\mu)| < \alpha^2|\phi_0(\mu)|. \tag{4.480}$$

As the square $S_k$, $k \in \mathbb{N}$ is a closed curve, then it follows from (4.480) and Rouché's theorem that the functions $\phi_0$ and $\phi$ have the same number of zeros inside the square $S_k$.

**Proposition 4.53.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2)*, where $B_1(y) = y''(0)$, $B_2(y) = y^{(3)}(0)$, $B_3y = y'(a) - i\alpha\lambda y''(a)$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a)$, counted with multiplicity, can be enumerated in such* *a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{cases} \hat{\mu}_k^{\pm} & = \pm(2k-5)\frac{\pi}{2a} + o(1), \quad if \ k > 0, \\ \hat{\mu}_k^{\pm} & = \pm i(2|k|-5)\frac{\pi}{2a} + o(1), \quad if \ k < 0. \end{cases}$$

*In particular, there is an even number of pure imaginary eigenvalues.*

**Remark 4.54.** We give the enumeration of the zeros of $\phi$ inside the square $S_{k_1}$, where $k_1 = \max\{\hat{m}_0, \hat{m}_1\}$. The remainder of the proof is identical to the remainder of the proof derived for Proposition 4.25.

*Proof.* The function $\phi$ has $4k_1$ number of zeros inside the square $S_{k_1}$. These zeros are the following

$$\hat{\mu}_1^{\pm} = \pm\left(-\frac{3\pi}{2a}\right) + o(1),\ \hat{\mu}_2^{\pm} = \pm\left(-\frac{\pi}{2a}\right) + o(1), \ldots, \hat{\mu}_{k_1}^{\pm} = \pm(2k_1 - 5)\frac{\pi}{2a} + o(1),$$
$$\hat{\mu}_{-1}^{\pm} = \pm i\left(-\frac{3\pi}{2a}\right) + o(1),\ \hat{\mu}_{-2}^{\pm} = \pm i\left(-\frac{\pi}{2a}\right) + o(1), \ldots, \hat{\mu}_{-(k_1)}^{\pm} = \pm i(2k_1 - 5)\frac{\pi}{2a} + o(1).$$

Thus

$$\hat{\mu}_k^{\pm} = \pm(2k-5)\frac{\pi}{2a} + o(1), \text{ where } 1 \leq k \leq k_1,$$
$$\hat{\mu}_{-k}^{\pm} = \pm i(2|k|-5)\frac{\pi}{2a} + o(1), \text{ where } -k_1 \leq k \leq -1.$$

Using the approach of the proof of Proposition 4.25, we can show that the zeros of $\phi$ for $k \geq k_1$ are

$$\hat{\mu}_k^{\pm},\ k = k_1, k_1+1, \ldots$$
$$-k_1, -k_1-1, \ldots$$

and satisfy

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(2k-5)\frac{\pi}{2a} + o(1), \text{ where } k \geq k_1, \\ \hat{\mu}_{-k}^{\pm} = \pm i(2|k|-5)\frac{\pi}{2a} + o(1), \text{ where } k \leq -k_1, \end{cases}$$

see (4.475). $\square$

## 4.7 The boundary terms $B_1y$ and $B_2y$ are the following: $B_1y = y'(0)$ and $B_2y = y^{(3)}(0)$

Using the canonical fundamental system, then

$$\begin{cases} B_1y_1 = y_1'(0) = 0, \\ B_1y_2 = y_2'(0) = 1, \\ B_1y_3 = y_3'(0) = 0, \\ B_1y_4 = y_4'(0) = 0, \end{cases} \qquad \begin{cases} B_2y_1 = y_1^{(3)}(0) = 0, \\ B_2y_2 = y_2^{(3)}(0) = 0, \\ B_2y_3 = y_3^{(3)}(0) = 0, \\ B_2y_4 = y_4^{(3)}(0) = 1. \end{cases}$$

It follows that the characteristic matrix of this particular boundary problem is

$$M_c = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ B_3y_1 & B_3y_2 & B_3y_3 & B_3y_4 \\ B_4y_1 & B_4y_2 & B_4y_3 & B_4y_4 \end{pmatrix}. \tag{4.481}$$

The determinant of the characteristic matrix $M_c$ gives the characteristic function of the differential equation (3.2). The shape of the matrix $M_c$ leads to a reduced characteristic matrix of the boundary value problem.

The reduced characteristic matrix of the boundary value problem is

$$M = \begin{pmatrix} B_3 \\ B_4 \end{pmatrix} \begin{pmatrix} y_1 & y_3 \end{pmatrix} = \begin{pmatrix} B_3y_1 & B_3y_3 \\ B_4y_1 & B_4y_3 \end{pmatrix}. \tag{4.482}$$

It is easy to check that $\det M_c = -\det M$.

### 4.7.1 Asymptotic of the eigenvalues for $B_3 y = y''(a) + i\alpha\lambda y'(a)$ and $B_4 y = y^{(3)}(a) - i\alpha\lambda y(a)$

It follows from (4.482) that

$$\begin{aligned}
\det M &= B_3 y_1 B_4 y_3 - B_4 y_1 B_3 y_3 \\
&= (y_1''(a) + i\alpha\mu^2 y_1'(a))(y_3^{(3)}(a) - i\alpha\mu^2 y_3(a)) - (y_1^{(3)}(a) - i\alpha\mu^2 y_1(a))(y_3''(a) + i\alpha\mu^2 y_3'(a)) \\
&= y_1''(a) y_3^{(3)}(a) + \alpha^2\mu^4 y_1'(a) y_3(a) - y_1^{(3)}(a) y_3''(a) - \alpha^2\mu^4 y_1(a) y_3'(a) + i\alpha\mu^2 (y_1'(a) y_3^{(3)}(a) \\
&- y_1''(a) y_3(a) - y_1^{(3)}(a) y_3'(a) + y_1(a) y_3''(a)) \\
&= y_1''(a) y_3^{(3)}(a) - y_1^{(3)}(a) y_3''(a) + \alpha^2\mu^4 y_1'(a) y_3(a) - \alpha^2\mu^4 y_1(a) y_3'(a) \\
&+ i\alpha\mu^2 (y_1'(a) y_3^{(3)}(a) - y_1''(a) y_3(a) + y_1(a) y_3''(a) - y_3'(a) y_1^{(3)}(a)).
\end{aligned} \tag{4.483}$$

Using (4.361) and (4.209), we have

$$\begin{aligned}
\det M &= \mu^8 y_4'(a) y_4(a) - \mu^4 y_4''(a) y_4^{(3)}(a) + \alpha^2\mu^8 y_4(a) y_4'(a) - \alpha^2\mu^4 y_4^{(3)}(a) y_4''(a) \\
&+ i\alpha\mu^2 (\mu^8 (y_4(a))^2 - \mu^4 (y_4'(a))^2 + (y_4^{(3)}(a))^2 - \mu^4 (y_4''(a))^2 \\
&= (1+\alpha^2)\mu^8 y_4(a) y_4'(a) - (1+\alpha^2)\mu^4 y_4''(a) y_4^{(3)}(a) \\
&+ i\alpha\mu^2 (\mu^8 (y_4(a))^2 + \mu^4 (y_4'(a))^2 - (y_4^{(3)}(a))^2 - \mu^4 (y_4''(a))^2).
\end{aligned} \tag{4.484}$$

Let

$$A_1(a) = (1+\alpha^2)\mu^8 y_4(a) y_4'(a) - (1+\alpha^2)\mu^4 y_4''(a) y_4^{(3)}(a) \tag{4.485}$$

$$A_2(a) = \mu^8 (y_4(a))^2 - \mu^4 (y_4'(a))^2 + (y_4^{(3)}(a))^2 - \mu^4 (y_4''(a))^2. \tag{4.486}$$

It follows from (4.212) and (4.213), that

$$\begin{aligned}
A_1(a) &= (1+\alpha^2)\mu^8 \left( \frac{1}{4\mu^5} \sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^5} \sin(\mu a)\cosh(\mu a) - \frac{1}{4\mu^5} \sinh(\mu a)\cos(\mu a) \right. \\
&\left. + \frac{1}{4\mu^5} \sinh(\mu a)\cosh(\mu a) \right) - (1+\alpha^2)\mu^4 \left( \frac{1}{4\mu} \sin(\mu a)\cos(\mu a) + \frac{1}{4\mu} \sinh(\mu a)\cos(\mu a) \right. \\
&\left. + \frac{1}{4\mu} \sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu} \sinh(\mu a)\cosh(\mu a) \right) \\
&= -(1+\alpha^2)\frac{\mu^3}{2} (\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)),
\end{aligned} \tag{4.487}$$

while (4.30), (4.31), (4.32) and (4.33) give

$$\begin{aligned}
A_2(a) = \mu^8 & \left(\frac{1}{4\mu^6}\sin^2(\mu a) - \frac{1}{2\mu^6}\sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^6}\sinh^2(\mu a)\right) \\
& - \mu^4 \left(\frac{1}{4\mu^4}\cos^2(\mu a) - \frac{1}{2\mu^4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4\mu^4}\cosh^2(\mu a)\right) \\
& - \mu^4 \left(\frac{1}{4\mu^2}\sin^2(\mu a) + \frac{1}{2\mu^2}\sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^2}\sinh^2(\mu a)\right) \\
& + \frac{1}{4}\cos^2(\mu a) + \frac{2}{4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4}\cosh^2(\mu a) \\
& = -\mu^2 \sin(\mu a)\sinh(\mu a) + \cos(\mu a)\cosh(\mu a). \qquad (4.488)
\end{aligned}$$

Putting (4.484), (4.485), (4.486), (4.487) and (4.488) together, we have

$$\begin{aligned}
\det M &= -i\alpha\mu^4 \sin(\mu a)\sinh(\mu a) + i\alpha\mu^2\cos(\mu a)\cosh(\mu a) \\
&\quad - (1+\alpha^2)\frac{\mu^3}{2}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)). \qquad (4.489)
\end{aligned}$$

Thus the characteristic equation $2i\det M = 0$ is

$$\phi(\mu) := 2\alpha\phi_0(\mu) + \phi_1(\mu) = 0, \qquad (4.490)$$

where

$$\phi_0(\mu) = \mu^4 \sin(\mu a)\sinh(\mu a), \qquad (4.491)$$

$$\begin{aligned}
\phi_1(\mu) &= -2\alpha\mu^2\cos(\mu a)\cosh(\mu a) \\
&\quad - i(1+\alpha^2)\mu^3(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)). \qquad (4.492)
\end{aligned}$$

The zeros of the function $\phi_0$ are 0, the zeros of $\sin(\mu a)$ and the zeros of $\sinh(\mu a)$. The zeros of $\sin(\mu a)$ are $\pm(k-1)\frac{\pi}{a}$, $k = 2, 3, \ldots$. Since $\sin(i\mu a) = i\sinh(\mu a)$ and $\sinh(i\mu a) = i\sin(\mu a)$, then the zeros of $\sinh(\mu a)$ are $\pm i(k-1)\frac{\pi}{a}$, where $k = 2, 3 \ldots$. Thus the zeros of $\phi_0$ are

$$0,\ \mu_k^{\pm} = \pm(k-1)\frac{\pi}{a},\ \hat{\mu}_{-k}^{\pm} = \pm i(k-1)\frac{\pi}{a},\ k = 2, 3, \ldots.$$

We recall that 0 is a zero of multiplicity 4 of $\mu \mapsto \mu^4$, while it is simple zero of $\mu \mapsto \sin(\mu a)$ and of $\mu \mapsto \sinh(\mu a)$. Thus 0 is a zero of multiplicity 6 of $\phi_0$. Whence the zeros of $\phi_0$, counted with multiplicity, are

$$\left.\begin{aligned}
&\mu_{-1}^{\pm} = 0,\ \mu_0^{\pm} = 0,\ \mu_1^{\pm} = 0,\ \ \mu_k^{\pm} = \pm(k-1)\tfrac{\pi}{a}, \\
&\mu_{-k}^{\pm} = \pm i(k-1)\tfrac{\pi}{a},\ k = 2, 3, \ldots
\end{aligned}\right\}. \qquad (4.493)$$

Let

$$\left.\begin{array}{l} R_k \text{ be the rectangles with vertices } k\frac{\pi}{a} \pm \varepsilon \pm i\varepsilon,\ k \in \mathbb{N} \text{ and } \varepsilon \in (0, \frac{\pi}{2a}),\ R_{-k} \\ \text{the symmetric image of } R_k \text{ with respet to the imaginary axis}, \widetilde{R}_k \text{ and } \widetilde{R}_{-k} \\ \text{the respective images of the rectangles } R_k \text{ and } R_{-k} \text{ by the rotation of angle} \\ \frac{\pi}{2} \end{array}\right\}. \tag{4.494}$$

Since $\varepsilon < \frac{\pi}{2a}$, then the rectangles $R_k$, $k \in \mathbb{N}$ do not intersect. We know that for $\mu \neq k\frac{\pi}{a}$, $k \in \mathbb{N}$, the function $\mu \mapsto \cot(\mu a)$ is periodic of period $\frac{\pi}{a}$ and it is also continuous. As the rectangles $R_k$, $k \in \mathbb{N}$ are closed curves, then $\cot(\mu a)$ is bounded on the rectangles $R_k$. Because the function $\mu \to \cot(\mu a)$ is periodic of period $\frac{\pi}{a}$, then there exists $\beta_1 > 0$ such that for all $\mu$ on the rectangles $R_k$, $k \in \mathbb{N}$,

$$|\cos(\mu a)| \leq \beta_1 |\sin(\mu a)|. \tag{4.495}$$

we have the same estimate for all $\mu$ on the rectangles $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{N}$. It follows from (4.155) that there exist $\beta_2 \geq 1$ and $j_0 > 0$ such that for all $\mu$ on the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{N}$ and $k \geq j_0$,

$$|\cosh(\mu a)| \leq \beta_2 |\sinh(\mu a)|. \tag{4.496}$$

Let

$$\phi_{10}(\mu) = -2\alpha\mu^2 \cos(\mu a)\cosh(\mu a), \tag{4.497}$$

$$\phi_{11}(\mu) = -i(1+\alpha^2)\mu^3(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)). \tag{4.498}$$

It follows from (4.495) and (4.496) that there exists $k_0 = \frac{a}{\pi}\sqrt{3\beta_1\beta_2}$, such that for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq m_0 = \max\{j_0, k_0\}$,

$$\begin{aligned} |\phi_{10}(\mu)| &= 2\alpha|\mu|^2 |\cos(\mu a)\cosh(\mu a)| \\ &\leq 2\alpha|\mu|^2 \beta_1\beta_2 |\sin(\mu a)\sinh(\mu a)| \\ &= 2\beta_1\beta_2\alpha|\mu|^2 |\sin(\mu a)\sinh(\mu a)| \\ &= \frac{2\beta_1\beta_2}{|\mu|^2}\alpha|\mu|^4 |\sin(\mu a)\sinh(\mu a)| \\ &< \alpha|\mu|^4 |\sin(\mu a)\sinh(\mu a)| = \frac{\alpha}{2}|\phi_0(\mu)|, \end{aligned} \tag{4.499}$$

while there exists $\widetilde{k}_0 = \frac{2a(1+\alpha^2)(\beta_1+\beta_2)}{\alpha\pi}$, such that for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_0 = \max\{j_0, \widetilde{k}_0\}$,

$$\begin{aligned}|\phi_{11}(\mu)| &= |1+\alpha^2||\mu|^3|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\ &< \frac{1}{\alpha}(1+\alpha^2)(\beta_1+\beta_2)\alpha|\mu|^3|\sin(\mu a)\sinh(\mu a)| \\ &< \frac{(1+\alpha^2)(\beta_1+\beta_2)}{\alpha|\mu|}\alpha|\mu|^4|\sin(\mu a)\sinh(\mu a)| \\ &< \alpha|\mu|^4|\sin(\mu a)\sinh(\mu a)| = \frac{\alpha}{2}|\phi_0(\mu)|. \end{aligned} \tag{4.500}$$

Putting (4.499) and (4.500) together, then we have for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_0 = \max\{m_0, \widetilde{m}_0\}$,

$$|\phi_1(\mu)| \leq |\phi_{10}(\mu)| + |\phi_{11}(\mu)| < \alpha|\phi_0(\mu)|. \tag{4.501}$$

Since the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, $k \in \mathbb{N}$ are closed curves, then it follows from (4.490), (4.491), (4.492), (4.493), (4.501) and Rouché's theorem that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$, where the asymptotics of the zeros of $\phi_0$ are

$$\left.\begin{aligned}&\hat{\mu}_k = \pm(k-1)\tfrac{\pi}{a} + o(1), \text{ where } k \geq \hat{m}_0 \text{ and} \\ &\hat{\mu}_k = \pm i(|k|-1)\tfrac{\pi}{a} + o(1) \text{ where } k \leq -\hat{m}_0\end{aligned}\right\}, \tag{4.502}$$

where $o(1) \to 0$ as $|k| \to \infty$.

Let

$$S_k \text{ be the square with vertices } \pm(2k+1)\frac{\pi}{2a} \pm i(2k+1)\frac{\pi}{2a},\ k \in \mathbb{N}. \tag{4.503}$$

We have for $\mu = (2k+1)\frac{\pi}{2a} + i\gamma$, $k \in \mathbb{N}$ and $\gamma \in \mathbb{R}$

$$\begin{aligned}\cot(\mu a) &= \cot\left(\left((2k+1)\frac{\pi}{2a} + i\gamma\right)a\right) = \cot\left(k\pi + i\gamma a + \frac{\pi}{2}\right) \\ &= -\tan(k\pi + i\gamma a) = -\tan(i\gamma a) = -i\tanh(\gamma a).\end{aligned} \tag{4.504}$$

Since $|\tanh(\gamma a)| \leq 1$ for all $\gamma \in \mathbb{R}$, then (4.504) implies that there exists $\delta_1 > 0$ such that for all $\mu = (2k+1)\frac{\pi}{2a} + i\gamma$, $k \in \mathbb{N}$ and $\gamma \in \mathbb{R}$

$$|\cos(\mu a)| \leq \delta_1|\sin(\mu a)|. \tag{4.505}$$

It follows from (4.155) that there exist $\delta_2 \geq 1$ and $j_1 > 0$ such that for all $\mu = (2k+1)\frac{\pi}{2a} + i\gamma$, where $\gamma \in \mathbb{R}$ and $k \in \mathbb{N}$ with $k \geq j_1$

$$|\cosh(\mu a)| \leq \delta_2 |\sinh(\mu a)|. \tag{4.506}$$

Interchanging cot and coth, we obtain for $\mu = \gamma + i(2k+1)\frac{\pi}{2a}$

$$\begin{aligned}
\coth\left(\left(\gamma + i(2k+1)\frac{\pi}{2a}\right)a\right) &= \coth\left(\gamma a + i\left(k\pi + \frac{\pi}{2}\right)\right) = \coth\left(i\left(-i\gamma a + k\pi + \frac{\pi}{2}\right)\right) \\
&= -i\cot\left(-i\gamma a + k\pi + \frac{\pi}{2}\right) = i\tan(-i\gamma a + k\pi) \\
&= i\tan(-i\gamma a) = -i\tan(i\gamma a) = \tanh(\gamma a).
\end{aligned} \tag{4.507}$$

It results from (4.49), (4.504) and (4.507) that

$$\cot(\mu a) = \cot\left(\left(\gamma + i(2k+1)\frac{\pi}{2a}\right)a\right) \to \pm 1 \text{ uniformly in } \gamma \text{ as } k \to \infty. \tag{4.508}$$

Thus we have the same estimates (4.505) and (4.506) for $\mu = \gamma + i(2k+1)\frac{\pi}{2a}$, where $\gamma \in \mathbb{R}$ and $k \in \mathbb{N}$, with $k \geq j_1$. Whence for all $\mu$ on the square $S_k$ with vertices $(2k+1)\frac{\pi}{2a} \pm i(2k+1)\frac{\pi}{2a}$,

$$|\cos(\mu a)| \leq \delta_1 |\sin(\mu a)|, \tag{4.509}$$

while for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq j_1$

$$|\cos(\mu a)| \leq \delta_2 |\sin(\mu a)|. \tag{4.510}$$

It follows from (4.509) and (4.510) that there exists $k_1 = \frac{a}{\pi}\sqrt{3\delta_1\delta_2}$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq m_1 = \max\{k_1, j_1\}$,

$$\begin{aligned}
|\phi_{10}(\mu)| &= 2\alpha|\mu|^2 |\cos(\mu a)\cosh(\mu a)| \\
&\leq 2\alpha|\mu|^2 \delta_1\delta_2 |\sin(\mu a)\sinh(\mu a)| \\
&= \frac{2\delta_1\delta_2}{|\mu|^2}\alpha|\mu|^4 |\sin(\mu a)\sinh(\mu a)| \\
&< \frac{\alpha}{2}|\mu|^4 |\sin(\mu a)\sinh(\mu a)| = \frac{\alpha}{2}|\phi_0(\mu)|,
\end{aligned} \tag{4.511}$$

while there exists $\widetilde{k}_1 = \frac{2a(1+\alpha^2)(\delta_1+\delta_2)}{\alpha\pi}$, such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and

$k \geq \widetilde{m}_1 = \max\{j_1, \widetilde{k}_1\}$,

$$\begin{aligned}
|\phi_{11}(\mu)| &= |1+\alpha^2||\mu|^3|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\
&\leq \frac{1}{\alpha}(1+\alpha^2)(\delta_1+\delta_2)\alpha|\mu|^3|\sin(\mu a)\sinh(\mu a)| \\
&= \frac{(1+\alpha^2)(\beta_1+\beta_2)}{\alpha|\mu|}\alpha|\mu|^4|\sin(\mu a)\sinh(\mu a)| \\
&< \alpha|\mu|^4|\sin(\mu a)\sinh(\mu a)| = \frac{\alpha}{2}|\phi_0(\mu)|. \qquad (4.512)
\end{aligned}$$

Putting (4.511) and (4.512) together, we have for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_1 = \max\{m_1, \widetilde{m}_1\}$

$$|\phi_1(\mu)| \leq |\phi_{10}(\mu)| + |\phi_{11}(\mu)| < \alpha|\phi_0(\mu)|. \qquad (4.513)$$

Since the square $S_k$, $k \in \mathbb{N}$ is closed curve, then it follows from (4.491), (4.492), (4.493) and (4.511) and Rouché's theorem that the functions $\phi_0$ and $\phi$ have the same number of zeros inside the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_1$.

**Remark 4.55.** Let $k_1 = \max\{\hat{m}_0, \hat{m}_1\}$. As 0 is a zero of multiplicity 6 of $\phi_0$, $\mu_k^{\pm}$ and $\mu_{-k}^{\pm}$, $k \in \mathbb{N}$ its simple zeros, then the number of zeros of $\phi_0$ and therefore of $\phi$ inside the square $S_k$, where $k \in \mathbb{N}$ and $k \geq k_1$ is $4k+2$.

**Proposition 4.56.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2)*, where $B_1(y) = y'(0)$, $B_2(y) = y^{(3)}(0)$, $B_3 y = y''(a) + i\alpha\lambda y'(a)$ and $B_4 y = y^{(3)}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}^{\pm})^2$ with*

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(k-1)\frac{\pi}{a} + o(1), & if\ k > 0, \\ \hat{\mu}_k^{\pm} = \pm i(|k|-1)\frac{\pi}{a} + o(1), & if\ k < 0. \end{cases}$$

*In particular, there is an odd number of pure imaginary eigenvalues.*

*Proof.* Let $D_{k+1}$ be the domain delimited by the squares $S_k$ and $S_{k+1}$, where $k \geq k_1$.

Using the same approach as in the proof of Proposition 4.23, we can prove that the number of zeros inside the the domain $D_{k+1}$, for $k \in \mathbb{N}$ and $k \geq k_1$ is exactly four.

The $4k_1 + 2$ zeros $\hat{\mu}_k^{\pm}$ of $\phi$ inside the square $S_{k_1}$ are the following

$$\hat{\mu}_0^{\pm} = \pm\left(-\tfrac{\pi}{a}\right) + o(1),\ \hat{\mu}_1^{\pm} = 0 + o(1), \ldots, \hat{\mu}_{k_1+1}^{\pm} = \pm\tfrac{k_1\pi}{a} + o(1),$$
$$\hat{\mu}_{-1}^{\pm} = 0 + o(1),\ \hat{\mu}_{-2}^{\pm} = \pm i\left(\tfrac{\pi}{a}\right) + o(1), \ldots, \hat{\mu}_{-(k_1+1)}^{\pm} = \pm i\tfrac{k_1\pi}{a} + o(1).$$

Thus

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(k-1)\frac{\pi}{a} + o(1), \text{ where } 0 \leq k \leq k_1 + 1 \text{ and} \\ \hat{\mu}_k^{\pm} = \pm i(|k|-1)\frac{\pi}{a} + o(1), \text{ where } -k_1 - 1 \leq k \leq -1. \end{cases}$$

The zeros of $\phi$ for $|k| \geq k_1$ are

$$\hat{\mu}_k^{\pm},\ k = k_1, k_1+1, \ldots$$
$$-k_1, -k_1 - 1, \ldots$$

and satisfy

$$\begin{cases} \hat{\mu}_k^{\pm} = \pm(k-1)\frac{\pi}{a} + o(1), \text{ where } k \geq k_1 \text{ and} \\ \hat{\mu}_k^{\pm} = \pm i(|k|-1)\frac{\pi}{a} + o(1), \text{ where } k \leq -k_1, \end{cases}$$

see (4.493).

We can observe that for the zeros $\mu$ of $\phi$ which are inside the square $S_k$ with vertices $\pm(2k+1)\frac{\pi}{2a} \pm i(2k+1)\frac{\pi}{2a}$, the numbers $\lambda = (\mu)^2$ are inside the curve $\mathscr{C}_k$, $k \in \mathbb{N}$ and $k \geq k_1$, given by the parametrization

$$\begin{cases} X = \pm((2k+1)\frac{\pi}{2a})^2 - \gamma^2 \\ Y = (2k+1)\gamma\frac{\pi}{a} \\ -(2k+1)\frac{\pi}{2a} \leq \gamma \leq (2k+1)\frac{\pi}{2a} \end{cases}. \tag{4.514}$$

The remainder of the proof of this proposition is similar to the remainder of the proof derived for Proposition 4.23. □

### 4.7.2 Asymptotic of the eigenvalues for $B_3 y = y''(a) + i\alpha\lambda y'(a)$ and $B_4 y = y(a) + i\alpha\lambda y^{(3)}(a)$

It follows from (4.482) that

$$\begin{aligned}
\det M &= B_3 y_1 B_4 y_3 - B_4 y_1 B_3 y_3 \\
&= (y_1''(a) + i\alpha\mu^2 y_1'(a))(y_3(a) + i\alpha\mu^2 y_3^{(3)}(a)) - (y_1(a) + i\alpha\mu^2 y_1^{(3)}(a))(y_3''(a) + i\alpha\mu^2 y_3'(a)) \\
&= y_1''(a)y_3(a) - \alpha^2\mu^4 y_1'(a)y_3^{(3)}(a) - y_1(a)y_3''(a) + \alpha^2\mu^4 y_1^{(3)}(a)y_3'(a) + i\alpha\mu^2(y_1'(a)y_3(a) \\
&+ y_1''(a)y_3^{(3)}(a) - y_1(a)y_3'(a) - y_1^{(3)}(a)y_3''(a)) \\
&= y_1''(a)y_3(a) - \alpha^2\mu^4 y_1'(a)y_3^{(3)}(a) - y_1(a)y_3''(a) + \alpha^2\mu^4 y_1^{(3)}(a)y_3'(a) \\
&+ i\alpha\mu^2(y_1'(a)y_3(a) + y_1''(a)y_3^{(3)}(a) - y_1(a)y_3'(a) - y_1^{(3)}(a)y_3''(a)).
\end{aligned} \tag{4.515}$$

It follows from (4.25) and (4.209) that

$$\begin{aligned}
\det M &= \mu^4(y_4'(a))^2 - \alpha^2\mu^{12}(y_4(a))^2 - (y_4^{(3)}(a))^2 + \alpha^2\mu^8(y_4''(a))^2 \\
&+ i\alpha\mu^2(\mu^4 y_4(a)y_4'(a) + \mu^8 y_4'(a)y_4(a) - y_4^{(3)}(a)y_4''(a) - \mu^4 y_4''(a)y_4^{(3)}(a)).
\end{aligned} \tag{4.516}$$

Let

$$A_1(a) = \mu^4(y_4'(a))^2 - \alpha^2\mu^{12}(y_4(a))^2 - (y_4^{(3)}(a))^2 + \alpha^2\mu^8(y_4''(a))^2 \tag{4.517}$$

$$A_2(a) = \mu^4 y_4(a)y_4'(a) + \mu^8 y_4'(a)y_4(a) - y_4^{(3)}(a)y_4''(a) - \mu^4 y_4''(a)y_4^{(3)}(a). \tag{4.518}$$

Then it follows from (4.30), (4.31), (4.32) and (4.33) that

$$\begin{aligned}
A_1(a) &= \mu^4\left(\frac{1}{4\mu^4}\cos^2(\mu a) - \frac{1}{2\mu^4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4\mu^4}\cosh^2(\mu a)\right) \\
&- \alpha^2\mu^{12}\left(\frac{1}{4\mu^6}\sin^2(\mu a) - \frac{1}{2\mu^6}\sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^6}\sinh^2(\mu a)\right) \\
&- \left(\frac{1}{4}\cos^2(\mu a) + \frac{2}{4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4}\cosh^2(\mu a)\right) \\
&+ \alpha^2\mu^8\left(\frac{1}{4\mu^2}\sin^2(\mu a) + \frac{1}{2\mu^2}\sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^2}\sinh^2(\mu a)\right) \\
&= \alpha^2\mu^6\sin(\mu a)\sinh(\mu a) - \cos(\mu a)\cosh(\mu a),
\end{aligned} \tag{4.519}$$

while (4.212) and (4.213) give

$$\begin{aligned}
A_2(a) &= \mu^4 \left( \frac{1}{4\mu^5} \sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^5} \sin(\mu a)\cosh(\mu a) - \frac{1}{4\mu^5} \sinh(\mu a)\cos(\mu a) \right. \\
&\quad \left. + \frac{1}{4\mu^5} \sinh(\mu a)\cosh(\mu a) \right) + \mu^8 \left( \frac{1}{4\mu^5} \sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^5} \sin(\mu a)\cosh(\mu a) \right. \\
&\quad \left. - \frac{1}{4\mu^5} \sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu^5} \sinh(\mu a)\cosh(\mu a) \right) - \left( \frac{1}{4\mu} \sin(\mu a)\cos(\mu a) \right. \\
&\quad \left. + \frac{1}{4\mu} \sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu} \sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu} \sinh(\mu a)\cosh(\mu a) \right) \\
&\quad - \mu^4 \left( \frac{1}{4\mu} \sin(\mu a)\cos(\mu a) + \frac{1}{4\mu} \sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu} \sin(\mu a)\cosh(\mu a) \right. \\
&\quad \left. + \frac{1}{4\mu} \sinh(\mu a)\cosh(\mu a) \right) \\
&= -\frac{\mu^3}{2} (\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \\
&\quad - \frac{1}{2\mu} (\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)). \qquad (4.520)
\end{aligned}$$

It follows from (4.516), (4.519) and (4.520) that

$$\begin{aligned}
\det M &= \alpha^2 \mu^6 \sin(\mu a)\sinh(\mu a) - \cos(\mu a)\cosh(\mu a) - \frac{i\alpha\mu^5}{2} (\sin(\mu a)\cosh(\mu a) \\
&\quad + \cos(\mu a)\sinh(\mu a)) - \frac{i\alpha\mu}{2} (\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a). \qquad (4.521)
\end{aligned}$$

Thus the characteristic equation $2 \det M = 0$ is

$$\phi(\mu) := 2\alpha^2 \phi_0(\mu) + \phi_1(\mu) = 0, \qquad (4.522)$$

where

$$\begin{aligned}
\phi_0(\mu) &= \mu^6 \sin(\mu a)\sinh(\mu a), \qquad (4.523) \\
\phi_1(\mu) &= -i\alpha\mu^5 (\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \\
&\quad - i\alpha\mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a) - 2\cos(\mu a)\cosh(\mu a). \qquad (4.524)
\end{aligned}$$

The zeros of $\phi_0$ are 0, the zeros of $\sin(\mu a)$ and the zeros of $\sinh(\mu a)$. But the zeros of $\sin(\mu a)$ are 0, $(k-2)\frac{\pi}{a}$, where $k = 3, 4 \ldots$. We recall that $\sinh(i\mu a) = i \sin(\mu a)$, while $\sin(i\mu a) = i \sinh(\mu a)$. Thus the zeros of $\sinh(\mu a)$ are $i(k-2)\frac{\pi}{a}$, where $k = 3, 4, \ldots$. Therefore the zeros of $\phi_0$ are

$$0, \ \mu_k^{\pm} = \pm(k-2)\frac{\pi}{a}, \ \mu_{-k}^{\pm} = \pm i(k-2)\frac{\pi}{a}, \text{ where } k = 3, 4 \ldots.$$

We recall that 0 is a zero of multiplicity 6 of $\mu \mapsto \mu^6$, while it is a simple zero of the function $\mu \mapsto \sin(\mu a)$ and $\mu \mapsto \sinh(\mu a)$. Hence 0 is a zero of multiplicity 8 of $\phi_0$ and the zeros of $\phi_0$ counted with multiplicity are

$$\left.\begin{array}{l} \mu_{-2}^{\pm} = 0,\ \mu_{-1}^{\pm} = 0,\ \mu_{1}^{\pm} = 0,\ \mu_{2}^{\pm} = 0, \\ \mu_{k}^{\pm} = \pm(k-2)\frac{\pi}{a},\ \mu_{-k}^{\pm} = \pm i(k-2)\frac{\pi}{a},\ k = 3, 4 \ldots \end{array}\right\}. \tag{4.525}$$

Let

$$\phi_{10}(\mu) = -i\alpha\mu^5(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)), \tag{4.526}$$

$$\phi_{11}(\mu) = -i\alpha\mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a), \tag{4.527}$$

$$\phi_{12}(\mu) = -2\cos(\mu a)\cosh(\mu a). \tag{4.528}$$

Let $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ be the rectangles defined in (4.494). Then it follows from (4.495) and (4.496) that there exists $k_0 = \frac{2a(\beta_1+\beta_2)}{\alpha\pi}$ such that for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq m_0 = \max\{j_0, k_0\}$,

$$\begin{aligned} |\phi_{10}(\mu)| &= \alpha|\mu|^5|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\ &\leq \frac{3}{2\alpha}(\beta_1+\beta_2)\frac{2\alpha^2}{3}|\mu|^5|\sin(\mu a)\sinh(\mu a)| \\ &= \frac{3(\beta_1+\beta_2)}{2\alpha|\mu|}\frac{2\alpha}{3}|\mu|^6|\sin(\mu a)\sinh(\mu a)| \\ &< \frac{2\alpha^2}{3}|\mu|^6|\sin(\mu a)\sinh(\mu a)| = \frac{2\alpha^2}{3}|\phi_0(\mu)|, \end{aligned} \tag{4.529}$$

while there exists $\widetilde{k}_0 = \frac{a}{\pi}\sqrt[5]{\frac{2(\beta_1+\beta_2)}{\alpha}}$, such that for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_0 = \max\{j_0, \widetilde{k}_0\}$

$$\begin{aligned} |\phi_{11}(\mu)| &= \alpha|\mu||\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\ &\leq \frac{3(\beta_1+\beta_2)}{2\alpha}\frac{2\alpha^2}{3}|\mu||\sin(\mu a)\sinh(\mu a)| \\ &= \frac{3(\beta_1+\beta_2)}{2\alpha|\mu|^5}\frac{2\alpha^2}{3}|\mu|^6|\sin(\mu a)\sinh(\mu a)| \\ &< \frac{2\alpha^2}{3}|\mu|^6|\sin(\mu a)\sinh(\mu a)| = \frac{2\alpha^2}{3}|\phi_0(\mu)|, \end{aligned} \tag{4.530}$$

finally there exists $\bar{k}_0 = \frac{a}{\pi}\sqrt[6]{\frac{4\beta_1\beta_2}{\alpha^2}}$ such that for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \bar{m}_0 = \max\{j_0, \bar{k}_0\}$

$$\begin{aligned}|\phi_{12}(\mu)| &= 2|\cos(\mu a)\cosh(\mu a)| \\ &\leq \frac{3\beta_1\beta_2}{\alpha^2|\mu|^6}\frac{2\alpha^2}{3}|\mu|^6|\sin(\mu a)\sinh(\mu a)| \\ &< \frac{2\alpha^2}{3}|\mu|^6|\sin(\mu a)\sinh(\mu a)| = \frac{2\alpha^2}{3}|\phi_0(\mu)|. \end{aligned} \tag{4.531}$$

Putting (4.529), (4.530) and (4.531) together, then for all $\mu$ on the rectangle $R_k$, $R_{-k}$, $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_0 = \max\{m_0, \widetilde{m}_0, \bar{m}_0\}$

$$|\phi_1(\mu)| \leq |\phi_{10}(\mu)| + |\phi_{11}(\mu)| + |\phi_{12}(\mu)| < 2\alpha^2|\phi_0(\mu)|. \tag{4.532}$$

Since the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, where $k \in \mathbb{N}$ are closed curves, then it follows from (4.522), (4.532) and Rouché's theorem that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$, where the asymptotics of the zeros of $\phi_0$ are

$$\left.\begin{aligned} \hat{\mu}_k^{\pm} &= \pm(k-2)\tfrac{\pi}{a} + o(1), \quad \text{where } k \geq \hat{m}_0 \text{ and} \\ \hat{\mu}_k^{\pm} &= \pm i(|k|-2)\tfrac{\pi}{a} + o(1), \quad \text{where } k \leq -\hat{m}_0 \end{aligned}\right\}, \tag{4.533}$$

where $o(1) \to 0$ as $|k| \to \infty$, see (4.525).

Let $S_k$, $k \in \mathbb{N}$ be the square defined in (4.503). Then it follows from (4.505) and (4.506) that there exists $\breve{k}_1 = \frac{2a(\delta_1+\delta_2)}{\alpha\pi}$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \breve{m}_1 = \max\{j_1, \breve{k}_1\}$,

$$\begin{aligned}|\phi_{10}(\mu)| &= \alpha|\mu|^5|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\ &\leq \frac{3}{2\alpha}(\delta_1+\delta_2)\frac{2\alpha^2}{3}|\mu|^5|\sin(\mu a)\sinh(\mu a)| \\ &< \frac{2\alpha^2}{3}|\mu|^6|\sin(\mu a)\sinh(\mu a)| = \frac{2\alpha^2}{3}|\phi_0(\mu)|, \end{aligned} \tag{4.534}$$

on the other hand there exists $\widetilde{k}_1 = \frac{a}{\pi}\sqrt[5]{\frac{2(\delta_1+\delta_2)}{\alpha}}$, such that for all $\mu$ on the square $S_k$, where

$k \in \mathbb{N}$ and $k \geq \widetilde{m}_1 = \max\{j_1, \widetilde{k}_1\}$

$$\begin{aligned}|\phi_{11}(\mu)| &= \alpha|\mu||\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\ &\leq \frac{3}{2\alpha}(\delta_1 + \delta_2)\frac{2\alpha^2}{3}|\mu||\sin(\mu a)\sinh(\mu a)| \\ &= \frac{3}{2\alpha|\mu|^5}(\delta_1 + \delta_2)\frac{2\alpha^2}{3}|\mu|^6|\sin(\mu a)\sinh(\mu a)| \\ &< \frac{2\alpha^2}{3}|\mu|^6|\sin(\mu a)\sinh(\mu a)| = \frac{2\alpha^2}{3}|\phi_0(\mu)|. \end{aligned} \tag{4.535}$$

Finally there exists $\bar{k}_1 = \frac{a}{\pi}\sqrt[6]{\frac{4\delta_1\delta_2}{\alpha^2}}$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \bar{m}_1 = \max\{j_1, \bar{k}_1\}$

$$\begin{aligned}|\phi_{12}(\mu)| &= 2|\cos(\mu a)\cosh(\mu a)| \\ &\leq \frac{3\delta_1\delta_2}{\alpha^2|\mu|^6}\frac{2\alpha^2}{3}|\mu|^6|\sin(\mu a)\sinh(\mu a)| \\ &< \frac{2\alpha^2}{3}|\mu|^6|\sin(\mu a)\sinh(\mu a)| = \frac{2\alpha^2}{3}|\phi_0(\mu)|. \end{aligned} \tag{4.536}$$

It follows from (4.534), (4.535) and (4.536) that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_1 = \max\{\check{m}_1, \widetilde{m}_1, \bar{m}_1\}$

$$|\phi_1(\mu)| \leq |\phi_{10}(\mu)| + |\phi_{11}(\mu)| + |\phi_{12}(\mu)| < 2\alpha^2|\phi_0(\mu)|. \tag{4.537}$$

Since the square $S_k$ is a closed curve, then (4.537) and Rouché's theorem imply that $\phi$ and $\phi_0$ have the same number of zeros inside the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_1$.

**Remark 4.57.** Let $k_1 = \max\{\hat{m}_1, \hat{m}_0\}$. Since 0 is a zero of multiplicity 8 of $\phi_0$, $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$ are its simple zeros, then the number of zeros of $\phi_0$ and therefore of $\phi$ inside the square $S_k$, where $k \in \mathbb{N}$ and $k \geq k_1$ is $4k$. Thus we have the following result.

**Proposition 4.58.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y'(0)$, $B_2(y) = y^{(3)}(0)$, $B_3y = y''(a) + i\alpha\lambda y'(a)$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}^{\pm})^2$ with*

$$\begin{cases}\hat{\mu}_k^{\pm} = \pm(k-2)\frac{\pi}{a} + o(1), & if\ k > 0, \\ \hat{\mu}_k^{\pm} = \pm i(|k|-2)\frac{\pi}{a} + o(1), & if\ k < 0.\end{cases}$$

*In particular, there is an even number of pure imaginary eigenvalues.*

*Proof.* Let $D_{k+1}$ the domain delimited by the squares $S_k$ and $S_{k+1}$, where $k \geq k_1$. In a similar manner as in the proof of Proposition 4.23, we can prove that the number of zeros inside the the domain $D_{k+1}$, for $k \in \mathbb{N}$ and $k \geq \hat{k}_1$ is exactly four.

The $4k_1$ zeros $\hat{\mu}_k^{\pm}$ of $\phi$ inside the square $S_{k_1}$ are the following

$$\begin{aligned}
&\hat{\mu}_1^{\pm} = \pm\left(-\tfrac{\pi}{a}\right) + o(1),\ \hat{\mu}_2^{\pm} = 0 + o(1), \ldots, \hat{\mu}_{k_1}^{\pm} = \pm\left(\tfrac{(k_1-2)\pi}{a}\right) + o(1),\\
&\hat{\mu}_{-1}^{\pm} = \pm i\left(-\tfrac{\pi}{a}\right) + o(1),\ \hat{\mu}_{-2}^{\pm} = 0 + o(1), \ldots, \hat{\mu}_{-(k_1)}^{\pm} = \pm i\left(\tfrac{(k_1-2)\pi}{a}\right) + o(1).
\end{aligned}$$

Thus

$$\begin{cases}
\hat{\mu}_k^{\pm} = \pm(k-2)\frac{\pi}{a} + o(1),\ \text{where } 0 \leq k \leq k_1 \text{ and}\\
\hat{\mu}_k^{\pm} = \pm i(|k|-2)\frac{\pi}{a} + o(1),\ \text{where } -k_1 \leq k \leq -1.
\end{cases}$$

The zeros of $\phi$ for $|k| \geq k_1$ are

$$\begin{aligned}
\hat{\mu}_k^{\pm},\ k = k_1, k_1, \ldots\\
-k_1, -k_1, \ldots
\end{aligned}$$

and satisfy

$$\begin{cases}
\hat{\mu}_k^{\pm} = \pm(k-2)\frac{\pi}{a} + o(1),\ \text{where } k \geq k_1 \text{ and}\\
\hat{\mu}_k^{\pm} = \pm i(|k|-2)\frac{\pi}{a} + o(1),\ \text{where } k \leq -k_1,
\end{cases}$$

see (4.493).

For the zeros $\mu$ of $\phi$ which are inside the square $S_k$ with vertices $\pm(2k+1)\frac{\pi}{2a} \pm i(2k+1)\frac{\pi}{2a}$, the numbers $\lambda = (\mu)^2$ are inside the curve $\mathscr{C}_k$ defined in (4.514), where $k \in \mathbb{N}$ and $k \geq k_1$.

The remainder of the proof of this proposition is similar to the remainder of the proof derived for Proposition 4.25. $\square$

### 4.7.3 Asymptotic of the eigenvalues for $B_3y = y'(a) - i\alpha\lambda y''(a)$ and $B_4y = y^{(3)}(a) - i\alpha\lambda y(a)$

It follows from (4.482) that

$$\begin{aligned}
\det M &= B_3y_1B_4y_3 - B_4y_1B_3y_3 \\
&= (y_1'(a) - i\alpha\mu^2 y_1''a))(y_3^{(3)}(a) - i\alpha\mu^2 y_3(a)) - (y_1^{(3)}(a) - i\alpha\mu^2 y_1(a))(y_3'(a) - i\alpha\mu^2 y_3''(a)) \\
&= y_1'(a)y_3^{(3)}(a) - \alpha^2\mu^4 y_1''(a)y_3(a) - y_1^{(3)}(a)y_3'(a) + \alpha^2\mu^4 y_1(a)y_3''(a) - i\alpha\mu^2(y_1''(a)y_3^{(3)}(a) \\
&+ y_1'(a)y_3(a) - y_1^{(3)}(a)y_3''(a) - y_1(a)y_3'(a)) \\
&= y_1'(a)y_3^{(3)}(a) - \alpha^2\mu^4 y_1''(a)y_3(a) - y_1^{(3)}(a)y_3'(a) + \alpha^2\mu^4 y_1(a)y_3''(a) \\
&- i\alpha\mu^2(y_1''(a)y_3^{(3)}(a) + y_1'(a)y_3(a) - y_1^{(3)}(a)y_3''(a) - y_1(a)y_3'(a)).
\end{aligned} \tag{4.538}$$

Thus (4.209) and (4.361) give

$$\begin{aligned}
\det M &= \mu^8(y_4(a))^2 - \alpha^2\mu^8(y_4'(a))^2 - \mu^4(y_4''(a))^2 + \alpha^2\mu^4(y_4^{(3)}(a))^2 \\
&- i\alpha\mu^2(\mu^8 y_4'(a)y_4(a) + \mu^4 y_4(a)y_4'(a) - \mu^4 y_4''(a)y_4^{(3)}(a) - y_4^{(3)}(a)y_4''(a)).
\end{aligned} \tag{4.539}$$

Let

$$A_1(a) = \mu^8(y_4(a))^2 - \alpha^2\mu^8(y_4'(a))^2 - \mu^4(y_4''(a))^2 + \alpha^2\mu^4(y_4^{(3)}(a))^2 \tag{4.540}$$

$$A_2(a) = \mu^8 y_4'(a)y_4(a) + \mu^4 y_4(a)y_4'(a) - \mu^4 y_4''(a)y_4^{(3)}(a) - y_4^{(3)}(a)y_4''(a). \tag{4.541}$$

Using (4.30), (4.31), (4.32) and (4.33), we have

$$\begin{aligned}
A_1(a) &= \mu^8\left(\frac{1}{4\mu^6}\sin^2(\mu a) - \frac{1}{2\mu^6}\sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^6}\sinh^2(\mu a)\right), \\
&- \alpha^2\mu^8\left(\frac{1}{4\mu^4}\cos^2(\mu a) - \frac{1}{2\mu^4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4\mu^4}\cosh^2(\mu a)\right) \\
&- \mu^4\left(\frac{1}{4\mu^2}\sin^2(\mu a) + \frac{1}{2\mu^2}\sin(\mu a)\sinh(\mu a) + \frac{1}{4\mu^2}\sinh^2(\mu a)\right) \\
&+ \alpha^2\mu^4\left(\frac{1}{4}\cos^2(\mu a) + \frac{2}{4}\cos(\mu a)\cosh(\mu a) + \frac{1}{4}\cosh^2(\mu a)\right) \\
&= \alpha^2\mu^4\cos(\mu a)\cosh(\mu a) - \mu^2\sin(\mu a)\sinh(\mu a),
\end{aligned} \tag{4.542}$$

while (4.212), and (4.213) give

$$
\begin{aligned}
A_2(a) = \mu^8 &\left(\frac{1}{4\mu^5}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^5}\sin(\mu a)\cosh(\mu a) - \frac{1}{4\mu^5}\sinh(\mu a)\cos(\mu a)\right.\\
&+ \frac{1}{4\mu^5}\sinh(\mu a)\cosh(\mu a)) + \mu^4\left(\frac{1}{4\mu^5}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^5}\sin(\mu a)\cosh(\mu a)\right.\\
&\left.- \frac{1}{4\mu^5}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu^5}\sinh(\mu a)\cosh(\mu a)\right) - \mu^4\left(\frac{1}{4\mu}\sin(\mu a)\cos(\mu a)\right.\\
&\left.+ \frac{1}{4\mu}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu}\sinh(\mu a)\cosh(\mu a)\right)\\
&- \left(\frac{1}{4\mu}\sin(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sin(\mu a)\cosh(\mu a)\right.\\
&\left.+ \frac{1}{4\mu}\sinh(\mu a)\cosh(\mu a)\right)\\
&= -\frac{\mu^3}{2}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a))\\
&- \frac{1}{2\mu}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)).
\end{aligned}
\tag{4.543}
$$

It follows from (4.539), (4.540), (4.541), (4.542) and (4.543) that

$$
\begin{aligned}
\det M = &\frac{i\alpha\mu^5}{2}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) + \alpha^2\mu^4\cos(\mu a)\cosh(\mu a)\\
&- \mu^2\sin(\mu a)\sinh(\mu a) + \frac{i\alpha\mu}{2}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)).
\end{aligned}
\tag{4.544}
$$

Hence the characteristic equation $-2i\det M = 0$ is

$$
\phi(\mu) := \alpha\phi_0(\mu) + \phi_1(\mu),
\tag{4.545}
$$

where

$$
\phi_0(\mu) = \mu^5(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a))
\tag{4.546}
$$

$$
\begin{aligned}
\phi_1(\mu) = &-2i\alpha^2\mu^4\cos(\mu a)\cosh(\mu a) + 2i\mu^2\sin(\mu a)\sinh(\mu a)\\
&+ \alpha\mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)).
\end{aligned}
\tag{4.547}
$$

The function $\phi_0$ defined in this subsection is identical to the function $\phi_0$ defined in (4.370), hence they have the same zeros. These zeros are the following

$$
0\ , \hat{\mu}_k^{\pm} = \pm(4k-5)\frac{\pi}{4a} + o(1),\ \hat{\mu}_{-k}^{\pm} = \pm i(4k-5)\frac{\pi}{4a} + o(1),\ k = 2, 3, \ldots,
\tag{4.548}
$$

see (4.372). We recall that 0 is a zero of multiplicity 6 of the function $\phi_0$ since 0 is a zero of multiplicity 1 for the function $\widetilde{\psi}_1(\mu) = \tan(\mu a) + \tanh(\mu a)$ defined in (4.227) and it is a zero of multiplicity 5 for the function $\mu \mapsto \psi_0(\mu) = \mu^5$. Whence the zeros of $\phi_0$ counted with multiplicity are the following

$$\left.\begin{array}{l}\hat{\mu}_{-1}^{\pm} = 0,\ \hat{\mu}_0^{\pm} = 0,\ \hat{\mu}_1^{\pm} = 0,\ \hat{\mu}_k^{\pm} = \pm(4k-5)\frac{\pi}{4a} + o(1),\\ \hat{\mu}_{-k}^{\pm} = \pm i(4k-5)\frac{\pi}{4a} + o(1),\ k = 2,3,4,\ldots\end{array}\right\}, \tag{4.549}$$

see (4.373). Let

$$\phi_{00}(\mu) = \cos(\mu a) + \sin(\mu a), \tag{4.550}$$

$$\phi_{01}(\mu) = (-1 + \tanh(\mu a))\cos(\mu a) - \frac{2i\alpha}{\mu}\cos(\mu a) + \frac{2i}{\alpha\mu^3}\sin(\mu a)\sinh(\mu a) + \frac{1}{\mu^4}(\sin(\mu a) + \cos(\mu a)\tanh(\mu a)). \tag{4.551}$$

Then

$$\phi_{02}(\mu) = \frac{\phi(\mu)}{\alpha\mu^5\cosh(\mu a)} = \phi_{00}(\mu) + \phi_{01}(\mu). \tag{4.552}$$

We recall that

$$\mu_k^{00} = \left(-\frac{\pi}{4a} \pm k\frac{\pi}{a}\right),\ k \in \mathbb{Z} \text{ are the zeros of } \phi_{00},$$

see (4.232) and

$$\widetilde{\mu}_k^{00} = i\left(-\frac{\pi}{4a} \pm k\frac{\pi}{a}\right),\ k \in \mathbb{Z} \text{ are the images of } \mu_k^{00} \text{ by the rotation of angle } \frac{\pi}{2},$$

see (4.233).

Let $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ be the rectangles defined in (4.234). We recall that $\mu_k^{00} = \left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in R_k$, $-\mu_k^{00} = -\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in R_{-k}$, $\widetilde{\mu}_k^{00} = i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_k$ and $-\widetilde{\mu}_k^{00} = -i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_{-k}$, see Remark 4.34 and the rectangles $R_k$ do not intersect, as well as the rectangles $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{Z}$, due to $\varepsilon < \frac{\pi}{2a}$. We recall also that there exists a constant $\rho(\varepsilon) > 0$ such that $|\phi_{00}(\mu)| > \rho(\varepsilon)$ for all $\mu$ on the rectangle $R_k$, $k \in \mathbb{Z}$, as $|\phi_{00}|$ is periodic of period $\frac{\pi}{a}$.

It follows from (4.235) and (4.236) that for all $\mu$ on the rectangle $R_k$, where $|k| > k_0(\varepsilon) = \max\{k_1(\varepsilon), k_2(\varepsilon)\}$ is sufficiently large positive, we have

$$|\phi_{01}(\mu)| < \frac{2\alpha\sqrt{2}}{|\mu|} + \frac{2\sqrt{2}}{\alpha|\mu|^3} + \frac{3\sqrt{2}}{|\mu|^4} + 3e^{-|\Re\mu a|}. \tag{4.553}$$

Since the right hand tends to 0 as $|\Re\mu a| \to \infty$, then for all $\mu$ on the rectangle $R_k$, where $k \in \mathbb{Z}$, $|k| \geq k_0(\varepsilon)$,

$$|\phi_{01}(\mu)| < |\phi_{00}(\mu)|. \tag{4.554}$$

For $\mu \in \mathbb{C}$, we have

$$\begin{aligned}
\phi_0(-\mu) &= (-\mu)^5(\sin(-\mu a)\cosh(-\mu a) + \sinh(-\mu a)\cos(-\mu a)) \\
&= -\mu^5(-\sin(\mu a)\cosh(\mu a) - \sinh(\mu a)\cos(\mu a)) \\
&= \mu^5(\sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cos(\mu a)) = \phi_0(\mu),
\end{aligned} \tag{4.555}$$

while

$$\begin{aligned}
\phi_1(-\mu) = &-2i\alpha^2(-\mu)^4\cos(-\mu a)\cosh(-\mu a) + 2i(-\mu)^2\sin(-\mu a)\sinh(-\mu a) \\
&+ \alpha(-\mu)(\sin(-\mu a)\cosh(-\mu a) + \cos(-\mu a)\sinh(-\mu a)) \\
= &-2i\alpha^2\mu^4\cos(\mu a)\cosh(\mu a) + 2i\mu^2\sin(\mu a)\sinh(\mu a) \\
&- \alpha\mu(-\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \\
= &-2i\alpha^2\mu^4\cos(\mu a)\cosh(\mu a) + 2i\mu^2\sin(\mu a)\sinh(\mu a) \\
&+ \alpha\mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) = \phi_1(\mu).
\end{aligned} \tag{4.556}$$

Thus the function $\phi_0$ and $\phi_1$ are even function and therefore $\phi$ is an even function. It follows that we have the same estimates (4.553) and (4.554) for all $\mu$ on the squares $R_k$ and $R_{-k}$, where $k \in \mathbb{Z}$, $|k| > k_0(\varepsilon)$ is large enough.

Let

$$\begin{aligned}
\widetilde{\phi}_0(\mu) = &\ \alpha\mu^5(\sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cos(\mu a)) \\
&+ \alpha\mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)),
\end{aligned} \tag{4.557}$$

$$\widetilde{\phi}_1(\mu) = -2i\alpha^2\mu^4\cos(\mu a)\cosh(\mu a) + 2i\mu^2\sin(\mu a)\sinh(\mu a). \tag{4.558}$$

Then

$$\phi(\mu) = \widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu). \tag{4.559}$$

For all $\mu \in \mathbb{C}$, we have

$$\begin{aligned}
\widetilde{\phi}_0(i\mu) &= \alpha(i\mu)^5(\sin(i\mu a)\cosh(i\mu a) + \sinh(i\mu a)\cos(i\mu a)) \\
&\quad + \alpha(i\mu)(\sin(i\mu a)\cosh(i\mu a) + \cos(i\mu a)\sinh(i\mu a)) \\
&= i\alpha\mu^5(i\sin(\mu a)\cosh(\mu a) + i\sinh(\mu a)\cos(\mu a)) \\
&\quad + i\alpha\mu(i\sinh(\mu a)\cos(\mu a) + i\cosh(\mu a)\sin(\mu a)) \\
&= -\alpha\mu^5(\sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cos(\mu a)) \\
&\quad - \alpha\mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) = -\widetilde{\phi}_0(\mu), \qquad (4.560)
\end{aligned}$$

while

$$\begin{aligned}
\widetilde{\phi}_1(i\mu) &= -2i\alpha^2(i\mu)^4\cos(i\mu a)\cosh(i\mu a) + 2i(i\mu)^2\sin(i\mu a)\sinh(i\mu a) \\
&= -2i\alpha^2\mu^4\cos(\mu a)\cosh(\mu a) + 2i(-\mu^2)(-\sin(\mu a)\sinh(\mu a)) \\
&= -2i\alpha^2\mu^4\cos(\mu a)\cosh(\mu a) + 2i\mu^2\sin(\mu a)\sinh(\mu a). \qquad (4.561)
\end{aligned}$$

Thus (4.560) and (4.561) imply

$$\phi(\mu) = -\widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu). \qquad (4.562)$$

It follows that $|\phi(\mu)|$ and $|\phi(i\mu)|$ have the same estimates $|\widetilde{\phi}_0(\mu)| + |\widetilde{\phi}_1(\mu)|$ for all $\mu$ on the rectangle $R_k$ or $R_{-k}$, where $k \in \mathbb{Z}$. As we can get the same estimates (4.553) and (4.554) for all $\mu$ on the rectangle $R_k$ or $R_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough, then we can have the same estimates for all $\mu$ on the rectangle $\widetilde{R}_k$ or $\widetilde{R}_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough.

Since the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{Z}$ are closed curves, then it follows from (4.550), (4.551), (4.552), (4.554) and Rouché's theorem that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$, where the asymptotics of the zeros of $\phi_0$ are

$$\left.\begin{aligned}
\hat{\mu}_k^{\pm} &= \pm(4k-5)\tfrac{\pi}{4a} + o(1), \text{ where } k \in \mathbb{N} \text{ and } k \geq k_0(\varepsilon) \\
\hat{\mu}_k^{\pm} &= \pm i(4|k|-5)\tfrac{\pi}{4a} + o(1), \text{ where } k \in \mathbb{N} \text{ and } k \leq -k_0(\varepsilon)
\end{aligned}\right\}, \qquad (4.563)$$

where $o(1) \to 0$ as $|k| \to \infty$, see (4.549).

Let $S_k$, $k \in \mathbb{N}$ be the square defined in (4.66) and

$$\widetilde{\phi}_2(\mu) = \coth(\mu a) + \cot(\mu a). \qquad (4.564)$$

It follows from (4.154), (4.155), (4.156), (4.159) and (4.160) that there exists $\widetilde{k}_0$, such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \widetilde{k}_0$

$$|\widetilde{\phi}_2(\mu)| \geq \frac{1}{2}. \tag{4.565}$$

Let

$$\phi_{10}(\mu) = \alpha\mu(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)), \tag{4.566}$$

$$\phi_{11}(\mu) = -2i\alpha^2\mu^4\cos(\mu a)\cosh(\mu a), \tag{4.567}$$

$$\phi_{12}(\mu) = 2i\mu^2\sin(\mu a)\sinh(\mu a). \tag{4.568}$$

Then there exists $m_1 = \frac{a}{\pi}\sqrt{2}$, such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq m_1$,

$$\begin{aligned} |\phi_{10}(\mu)| &= \alpha|\mu||\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\ &= \frac{3}{|\mu|^4}\frac{\alpha}{3}|\mu|^5|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| \\ &< \frac{\alpha}{3}|\mu|^5|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| = \frac{\alpha}{3}|\phi_0(\mu)|, \end{aligned} \tag{4.569}$$

while it follows from (4.227) and (4.249) there exists $\widetilde{k}_1 = \frac{13\alpha a}{\pi}$, such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_1 = \max\{\widetilde{k}_1, \hat{k}_0\}$,

$$\begin{aligned} \frac{\alpha}{3}|\phi_0(\mu)| &= \frac{\alpha}{3}|\mu|^5|\sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cos(\mu a)| \\ &= \frac{|\mu|}{6\alpha}|\tan(\mu a) + \tanh(\mu a)|2\alpha^2|\mu|^4|\cos(\mu a)\cosh(\mu a)| \\ &= \frac{|\mu|}{6\alpha}|\widetilde{\psi}_1(\mu)|2\alpha^2|\mu|^4|\cos(\mu a)\cosh(\mu a)| \\ &\geq \frac{|\mu|}{12\alpha}2\alpha^2|\mu|^4|\cos(\mu a)\cosh(\mu a)| = \frac{|\mu|}{12\alpha}|\phi_{11}(\mu)| > |\phi_{11}(\mu)|, \end{aligned} \tag{4.570}$$

finally it follows from (4.564) and (4.565) there exists $\bar{k}_1 = \frac{a}{\pi}\sqrt[3]{\frac{13}{\alpha}}$, such that for all $\mu$ on the square $S_k$, where $k \geq \bar{m}_1 = \max\{\widetilde{k}_0, \bar{k}_1\}$

$$\begin{aligned} \frac{\alpha}{3}|\phi_0(\mu)| &= \frac{\alpha}{3}|\mu|^5|\sin(\mu a)\cosh(\mu a) + \sinh(\mu a)\cos(\mu a)| \\ &= \frac{\alpha|\mu|^3}{6}|\cot(\mu a) + \coth(\mu a)|2|\mu|^2|\sin(\mu a)\sinh(\mu a)| \\ &= \frac{\alpha|\mu|^3}{6}|\widetilde{\phi}_2(\mu)||\phi_{12}(\mu)| \geq \frac{\alpha|\mu|^3}{12}|\phi_{12}(\mu)| > |\phi_{12}(\mu)|. \end{aligned} \tag{4.571}$$

Let $\hat{m}_1 = \max\{m_1, \widetilde{m}_1, \bar{m}_1\}$. Then it follows from (4.569), (4.570) and (4.571) that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_1$,

$$|\phi_1(\mu)| < \alpha|\phi_0(\mu)|. \tag{4.572}$$

As the square $S_k$, $k \in \mathbb{N}$ is a closed curve, then it follows from (4.572) and Rouché's theorem that $\phi$ and $\phi_0$ have the same number of zeros inside the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_1$.

**Remark 4.59.** Let $k_1 = \max\{k_0(\varepsilon), \hat{m}_1\}$. As 0 is a zero of multiplicity 6 of $\phi_0$, while $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$, where $k = 3, 4\ldots$, are its simple zeros, then the number of zeros of $\phi_0$ and therefore of $\phi$ inside the square $S_k$, where $k \in \mathbb{N}$ and $k \geq k_1$ is $4k + 2$. Thus we have the following proposition.

**Proposition 4.60.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y'(0)$, $B_2(y) = y^{(3)}(0)$, $B_3y = y'(a) - i\alpha\lambda y''(a)$ and $B_4y = y^{(3)}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{cases} \hat{\mu}_k^{\pm} & = \pm(4k-5)\frac{\pi}{4a} + o(1), \quad if\ k > 0, \\ \hat{\mu}_k^{\pm} & = \pm i(4|k|-5)\frac{\pi}{4a} + o(1), \quad if\ k < 0. \end{cases}$$

*In particular, there is an odd number of pure imaginary eigenvalues.*

**Remark 4.61.** The proof of Proposition 4.60 is identical to the proof of Proposition 4.46.

### 4.7.4 Asymptotic of the eigenvalues for $B_3y = y'(a) - i\alpha\lambda y''(a)$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a)$

It follows from (4.482) that

$$\begin{aligned}
\det M &= B_3y_1B_4y_3 - B_4y_1B_3y_3 \\
&= (y_1'(a) - i\alpha\mu^2 y_1''a))(y_3(a) + i\alpha\mu^2 y_3^{(3)}(a)) - (y_1(a) + i\alpha\mu^2 y_1^{(3)}(a))(y_3'(a) - i\alpha\mu^2 y_3''(a)) \\
&= y_1'(a)y_3(a) + \alpha^2\mu^4 y_1''(a)y_3^{(3)}(a) - y_1(a)y_3'(a) - \alpha^2\mu^4 y_1^{(3)}(a)y_3''(a) + i\alpha\mu^2(y_1'(a)y_3^{(3)}(a) \\
&- y_1''(a)y_3(a) + y_1(a)y_3''(a) - y_1^{(3)}(a)y_3'(a)) \\
&= y_1'(a)y_3(a) + \alpha^2\mu^4 y_1''(a)y_3^{(3)}(a) - y_1(a)y_3'(a) - \alpha^2\mu^4 y_1^{(3)}(a)y_3''(a) \\
&+ i\alpha\mu^2(y_1'(a)y_3^{(3)}(a) - y_1''(a)y_3(a) + y_1(a)y_3''(a) - y_1^{(3)}(a)y_3'(a)). \qquad (4.573)
\end{aligned}$$

Thus (4.209) and (4.361) give

$$\begin{aligned}
\det M &= \mu^4 y_4(a)y_4'(a) + \alpha^2\mu^{12}y_4'(a)y_4(a) - y_4^{(3)}(a)y_4''(a) - \alpha^2\mu^8 y_4''(a)y_4^{(3)}(a) \\
&+ i\alpha\mu^2(\mu^4(\mu^8(y_4(a))^2 - y_4'(a))^2 + (y_4^{(3)}(a))^2 - \mu^4(y_4''(a))^2). \qquad (4.574)
\end{aligned}$$

Let

$$A_1(a) = (\mu^4 + \alpha^2\mu^{12})y_4'(a)y_4(a) - (1 + \alpha^2\mu^8)y_4''(a)y_4^{(3)}(a), \qquad (4.575)$$

$$A_2(a) = \mu^8(y_4(a))^2 - \mu^4(y_4'(a))^2 + (y_4^{(3)}(a))^2 - \mu^4(y_4''(a))^2. \qquad (4.576)$$

Then it follows from (4.212) and (4.213), that

$$\begin{aligned}
A_1(a) &= \left(\mu^4 + \alpha^2\mu^{12}\right)\left(\frac{1}{4\mu^5}\sin(\mu a)\cos(\mu a) - \frac{1}{4\mu^5}\sin(\mu a)\cosh(\mu a)\right. \\
&\left.- \frac{1}{4\mu^5}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu^5}\sinh(\mu a)\cosh(\mu a)\right) - \left(1 + \alpha^2\mu^8\right)\left(\frac{1}{4\mu}\sin(\mu a)\cos(\mu a)\right. \\
&\left.+ \frac{1}{4\mu}\sinh(\mu a)\cos(\mu a) + \frac{1}{4\mu}\sin(\mu a)\cosh(\mu a) + \frac{1}{4\mu}\sinh(\mu a)\cosh(\mu a)\right) \\
&= -\frac{\alpha^2\mu^7}{2}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \\
&- \frac{1}{2\mu}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)), \qquad (4.577)
\end{aligned}$$

while (4.30), (4.31), (4.32) and (4.33) give

$$\begin{aligned}
A_2(a) &= \mu^8 \left( \frac{1}{4\mu^6} \sin^2(\mu a) - \frac{1}{2\mu^6} \sin(\mu a) \sinh(\mu a) + \frac{1}{4\mu^6} \sinh^2(\mu a) \right) \\
&\quad - \mu^4 \left( \frac{1}{4\mu^4} \cos^2(\mu a) - \frac{1}{2\mu^4} \cos(\mu a) \cosh(\mu a) + \frac{1}{4\mu^4} \cosh^2(\mu a) \right) \\
&\quad + \frac{1}{4} \cos^2(\mu a) + \frac{2}{4} \cos(\mu a) \cosh(\mu a) + \frac{1}{4} \cosh^2(\mu a) \\
&\quad - \mu^4 \left( \frac{1}{4\mu^2} \sin^2(\mu a) + \frac{1}{2\mu^2} \sin(\mu a) \sinh(\mu a) + \frac{1}{4\mu^2} \sinh^2(\mu a) \right) \\
&= -\mu^2 \sin(\mu a) \sinh(\mu a) + \cos(\mu a) \cosh(\mu a). \qquad (4.578)
\end{aligned}$$

Putting together (4.574), (4.575), (4.576), (4.577) and (4.578) we get

$$\begin{aligned}
\det M &= -\frac{\alpha^2\mu^7}{2} (\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) - i\alpha\mu^4 \sin(\mu a)\sinh(\mu a) \\
&\quad + i\alpha\mu^2 \cos(\mu a)\cosh(\mu a) - \frac{1}{2\mu} (\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)). \qquad (4.579)
\end{aligned}$$

It follows that the characteristic equation $-2\det M = 0$ is

$$\phi(\mu) := \alpha^2 \phi_0(\mu) + \phi_1(\mu a), \qquad (4.580)$$

where

$$\phi_0(\mu) = \mu^7 (\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)), \qquad (4.581)$$

$$\begin{aligned}
\phi_1(\mu) &= 2i\alpha\mu^4 \sin(\mu a)\sinh(\mu a) - 2i\alpha\mu^2 \cos(\mu a)\cosh(\mu a) \\
&\quad + \frac{1}{\mu} (\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)). \qquad (4.582)
\end{aligned}$$

The function $\phi_0$ defined in this subsection is identical to the function $\phi_0$ defined in (4.403). Thus the zeros of $\phi_0$ are

$$0 \ , \hat{\mu}_k^{\pm} = \pm(4k-9)\frac{\pi}{4a} + o(1), \ \hat{\mu}_{-k}^{\pm} = \pm i(4k-9)\frac{\pi}{4a} + o(1), \ k = 3, 4, \ldots,$$

see (4.405), while the zeros of $\phi_0$ counted with multiplicity are the following

$$\left. \begin{array}{l}
\hat{\mu}_{-2}^{\pm} = 0, \ \hat{\mu}_{-1}^{\pm} = 0, \ \hat{\mu}_1^{\pm} = 0, \ \hat{\mu}_2^{\pm} = 0, \\
\hat{\mu}_k^{\pm} = \pm(4k-9)\frac{\pi}{4a} + o(1), \\
\hat{\mu}_{-k}^{\pm} = \pm i(4k-9)\frac{\pi}{4a} + o(1), \ k = 3, 4, \ldots
\end{array} \right\},$$

see (4.406).

Let

$$\phi_{00}(\mu) = \cos(\mu a) + \sin(\mu a), \tag{4.583}$$

$$\begin{aligned}\phi_{01}(\mu) &= (-1 + \tanh(\mu a))\cos(\mu a) + \frac{2i}{\alpha\mu^3}\sin(\mu a)\tanh(\mu a) - \frac{2i}{\alpha\mu^5}\cos(\mu a) \\ &\quad + \frac{1}{\alpha^2\mu^8}(\sin(\mu a) + \cos(\mu a)\tanh(\mu a)).\end{aligned} \tag{4.584}$$

Then

$$\phi_{02}(\mu) = \frac{\phi(\mu)}{\alpha^2\mu^7\cosh(\mu a)} = \phi_{00}(\mu) + \phi_{01}(\mu). \tag{4.585}$$

We recall that the numbers

$$\mu_k^{00} = \left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right),\ k \in \mathbb{Z} \text{ are the zeros of } \phi_{00},$$

see (4.232) and

$$\widetilde{\mu}_k^{00} = i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right),\ k \in \mathbb{Z} \text{ are the images of } \mu_k^{00} \text{ by the rotation of angle } \frac{\pi}{2},$$

see (4.233).

Let $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ be the rectangles defined in (4.234). We have $\mu_k^{00} = \left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in R_k$, $-\mu_k^{00} = -\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in R_{-k}$, $\widetilde{\mu}_k^{00} = i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_k$ and $-\widetilde{\mu}_k^{00} = -i\left(-\frac{\pi}{4a} + k\frac{\pi}{a}\right) \in \widetilde{R}_{-k}$, see Remark 4.34. The rectangles $R_k$ do not intersect, as well as the rectangles $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$, $k \in \mathbb{Z}$, due to $\varepsilon < \frac{\pi}{2a}$. We recall that there exists a constant $\rho(\varepsilon) > 0$ such that $|\phi_{00}(\mu)| > \rho(\varepsilon)$ for all $\mu$ on the rectangle $R_k$, $k \in \mathbb{Z}$, as $|\phi_{00}|$ is periodic of period $\frac{\pi}{a}$.

It follows from (4.235) and (4.236) that for all $\mu$ on the rectangle $R_k$, where $|k| > k_0(\varepsilon) = \max\{k_1(\varepsilon), k_2(\varepsilon)\}$ is sufficiently large positive, we have

$$|\phi_{01}(\mu)| < \frac{2\sqrt{2}}{\alpha|\mu|^3} + \frac{2\sqrt{2}}{\alpha|\mu|^5} + \frac{3\sqrt{2}}{\alpha^2|\mu|^8} + 3e^{-|\Re\mu a|}. \tag{4.586}$$

Since the right hand tends to 0 as $|\Re\mu a| \to \infty$, then for all $\mu$ on the rectangle $R_k$, where $k \in \mathbb{Z}$, $|k| \geq k_0(\varepsilon)$,

$$|\phi_{01}(\mu)| < |\phi_{00}(\mu)|. \tag{4.587}$$

For all $\mu \in \mathbb{C}$, we have

$$\begin{aligned}\phi_0(-\mu) &= (-\mu)^7(\sin(-\mu a)\cosh(-\mu a) + \cos(-\mu a)\sinh(-\mu a)) \\ &= -\mu^7(-\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \\ &= \mu^7(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) = \phi_0(\mu), \end{aligned} \tag{4.588}$$

while

$$\begin{aligned}\phi_1(-\mu) &= 2i\alpha(-\mu)^4\sin(-\mu a)\sinh(-\mu a) - 2i\alpha(-\mu)^2\cos(-\mu a)\cosh(-\mu a) \\ &+ \frac{1}{-\mu}(\sin(-\mu a)\cosh(-\mu a) + \cos(-\mu a)\sinh(-\mu a)) \\ &= 2i\alpha\mu^4\sin(\mu a)\sinh(\mu a) - 2i\alpha\mu^2\cos(\mu a)\cosh(\mu a) \\ &+ \frac{-1}{\mu}(-\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)) \\ &= 2i\alpha\mu^4\sin(\mu a)\sinh(\mu a) - 2i\alpha\mu^2\cos(\mu a)\cosh(\mu a) \\ &+ \frac{1}{\mu}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) = \phi_1(\mu). \end{aligned} \tag{4.589}$$

It follows from (4.580), (4.588) and (4.589) that $\phi$ is an even function and therefore we have the same estimates (4.586) and (4.587) for all $\mu$ on the rectangle $R_{-k}$, where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$.

Let

$$\begin{aligned}\widetilde{\phi}_0(\mu) &= \alpha^2\mu^7(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \\ &+ \frac{1}{\mu}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \end{aligned} \tag{4.590}$$

$$\widetilde{\phi}_1(\mu) = 2i\alpha\mu^4\sin(\mu a)\sinh(\mu a) - 2i\alpha\mu^2\cos(\mu a)\cosh(\mu a). \tag{4.591}$$

Then

$$\phi(\mu) = \widetilde{\phi}_0(\mu) + \widetilde{\phi}_1(\mu) \tag{4.592}$$

and for all $\mu \in \mathbb{C}$

$$\begin{aligned}
\widetilde{\phi}_0(i\mu) &= \alpha^2(i\mu)^7(\sin(i\mu a)\cosh(i\mu a) + \cos(i\mu a)\sinh(i\mu a)) \\
&+ \frac{1}{i\mu}(\sin(i\mu a)\cosh(i\mu a) + \cos(i\mu a)\sinh(i\mu a)) \\
&= -\alpha^2 i\mu^7(i\sinh(\mu a)\cos(\mu a) + i\cosh(\mu a)\sin(\mu a)) \\
&+ \frac{1}{i\mu}(i\sinh(\mu a)\cos(\mu a) + i\cosh(\mu a)\sin(\mu a)) \\
&= \alpha^2\mu^7(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) \\
&+ \frac{1}{\mu}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)) = \widetilde{\phi}_0(\mu),
\end{aligned} \tag{4.593}$$

while

$$\begin{aligned}
\widetilde{\phi}_1(i\mu) &= 2i\alpha(i\mu)^4\sin(i\mu a)\sinh(i\mu a) - 2i\alpha(i\mu)^2\cos(i\mu a)\cosh(i\mu a) \\
&= -2i\alpha\mu^4\sin(\mu a)\sinh(\mu a) + 2i\alpha\mu^2\cos(\mu a)\cosh(\mu a) = -\widetilde{\phi}_1(\mu).
\end{aligned} \tag{4.594}$$

Thus

$$\phi(i\mu) = \widetilde{\phi}_0(\mu) - \widetilde{\phi}_1(\mu) \tag{4.595}$$

and $|\phi(\mu)|$ and $|\phi(i\mu)|$ have the same upper bound $|\widetilde{\phi}_0(\mu)| + |\widetilde{\phi}_1(\mu)|$ for all $\mu$ on the rectangles $R_k$ and $R_{-k}$, see (4.592) and (4.595). It follows from (4.580), (4.581) and (4.582) that we can obtain the same estimates (4.586), and (4.587) for all $\mu$ on the rectangles $R_k$ and $R_{-k}$ where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$. As $|\phi(\mu)|$ and $|\phi(i\mu)|$ have the same upper bound, then we can obtain the same estimates (4.586) and (4.587) for all $\mu$ on the rectangles $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ where $k \in \mathbb{Z}$ and $|k| \geq k_0(\varepsilon)$ large enough.

Since the rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$ and $\widetilde{R}_{-k}$ are closed curves, 0 is a zero of multiplicity 10 of $\phi_0$, while $\hat{\mu}_k^{\pm}$ and $\hat{\mu}_{-k}^{\pm}$, $k = 3, 4 \ldots$, are simple zeros, then it follows from (4.587) and Rouché's theorem that there are zeros of $\phi$ which have the same asymptotics as the zeros of $\phi_0$, where the asymptotics of the zeros of $\phi_0$ are

$$\left.\begin{array}{l}
\hat{\mu}_k^{\pm} = \pm(4k-9)\frac{\pi}{4a} + o(1), \text{ where } k \in \mathbb{Z}, k \geq k_0(\varepsilon) \text{ and} \\
\hat{\mu}_k^{\pm} = \pm i(4|k|-9)\frac{\pi}{4a} + o(1), \text{ where } k \in \mathbb{Z}, k \leq -k_0(\varepsilon)
\end{array}\right\}, \tag{4.596}$$

with $o(1) \to 0$ as $|k| \to \infty$, see (4.406).

Let $S_k$, $k \in \mathbb{N}$ be the square defined in (4.66) and

$$\phi_{10}(\mu) = \frac{1}{\mu}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)), \tag{4.597}$$

$$\phi_{11}(\mu) = 2i\alpha\mu^4 \sin(\mu a)\sinh(\mu a), \tag{4.598}$$

$$\phi_{12}(\mu) = -2i\alpha\mu^2 \cos(\mu a)\cosh(\mu a). \tag{4.599}$$

Then there exists $m_1 = \frac{a}{\pi}\sqrt[8]{\frac{4}{\alpha^2}}$ such that for all $\mu$ on the square $S_k$, where $k \geq m_1$,

$$\begin{aligned}|\phi_{10}(\mu)| &= \frac{1}{|\mu|}|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)|\\ &\frac{3}{\alpha^2|\mu|^8}\frac{\alpha^2}{3}|\mu|^7|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)|\\ &\leq \frac{\alpha^2}{3}|\mu|^7|\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)| = \frac{\alpha^2}{3}|\phi_0(\mu)|,\end{aligned} \tag{4.600}$$

while it follows from (4.564) and (4.565) that there exists $\widetilde{k}_1 = \frac{a}{\pi}\sqrt[3]{\frac{13}{\alpha}}$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \widetilde{m}_1 = \max\{\widetilde{k}_0, \widetilde{k}_1\}$

$$\begin{aligned}\frac{\alpha^2}{3}|\phi_0(\mu)| &= \frac{\alpha^2}{3}|\widetilde{\phi}_2(\mu)||\mu|^7|\sin(\mu a)\sinh(\mu a)|\\ &\geq \frac{\alpha|\mu|^3}{12}2\alpha|\mu|^4|\sin(\mu a)\sinh(\mu a)| = \frac{\alpha|\mu|^3}{12}|\phi_{11}(\mu)| > |\phi_{11}(\mu)|,\end{aligned} \tag{4.601}$$

finally it follows from (4.227) and (4.249) there exists $\bar{k}_1 = \frac{a}{\pi}\sqrt[5]{\frac{13}{\alpha}}$ such that for all $\mu$ on the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \bar{m}_1 = \max\{\widetilde{k}_0, \bar{k}_1\}$

$$\begin{aligned}\frac{\alpha^2}{3}|\phi_0(\mu)| &= \frac{\alpha^2}{3}|\widetilde{\psi}_1(\mu)||\mu|^7|\cos(\mu a)\cosh(\mu a)|\\ &\geq \frac{\alpha|\mu|^5}{12}2\alpha|\mu|^2|\cos(\mu a)\cosh(\mu a)| = \frac{\alpha|\mu|^5}{12}|\phi_{12}(\mu)| > |\phi_{12}(\mu)|.\end{aligned} \tag{4.602}$$

Let $\hat{m}_1 = \max\{m_1, \widetilde{m}_1, \bar{m}_1\}$. Then it follows from (4.600), (4.601) and (4.602) that for all $\mu$ on the square $S_k$ with vertices $\pm k\frac{\pi}{a} \pm ik\frac{\pi}{a}$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_1$,

$$|\phi_1(\mu)| < \alpha^2|\phi_0(\mu)|. \tag{4.603}$$

Since the square $S_k$, $k \in \mathbb{N}$ is a closed curve, then (4.603) and Rouché's theorem imply that $\phi_0$ and $\phi$ have the same number of zeros inside the square $S_k$, where $k \in \mathbb{N}$ and $k \geq \hat{m}_1$.

**Proposition 4.62.** *For $g = 0$, there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2)*, where $B_1(y) = y'(0)$, $B_2(y) = y^{(3)}(0)$, $B_3y = y'(a) -$*

*$i\alpha\lambda y''(a)$ and $B_4 y = y(a) + i\alpha\lambda y^{(3)}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{cases} \hat{\mu}_k^{\pm} & = \pm(4k-9)\frac{\pi}{4a} + o(1), \quad if\ k > 0, \\ \hat{\mu}_k^{\pm} & = \pm i(4|k|-9)\frac{\pi}{4a} + o(1), \quad if\ k < 0. \end{cases}$$

*In particular, there is an even number of pure imaginary eigenvalues.*

Since the function $\phi_0$ defined in (4.581) is identical to the function $\phi_0$ defined in (4.403), then Remark 4.49 and the proof of Proposition 4.48 are applicable for the above proposition.

# Chapter 5

# Asymptotics of eigenvalues of the self-adjoint fourth order differential operators

## 5.1 Introduction

We give in this chapter the spectral asymptotics of the eigenvalues of the problem (3.1)–(3.2), where $B_j(\lambda)y = y^{[p_j]}(0) = 0$ for $j = 1, 2$, while for $j = 3, 4$, $B_j(\lambda)y = y^{[p_j]}(a) + i\varepsilon_j\alpha\lambda y^{[q_j]}(a) = 0$ and $\varepsilon_j$ satisfies the conditions of Theorem 3.2. We start the chapter by presenting in Section 5.2 notions and properties of two-point boundary eigenvalue problems in $(L_2(a, b))^n$ in Subsection 5.2.1, followed in Subsection 5.2.2 by definitions and properties necessary for a better comprehension of Birkhoff regular problems that we give in Subsection 5.2.3. The notions and properties presented in Section 5.2 are used to investigate the Birkhoff regularities of the eigenvalue problems above mentioned. These Birkhoff regularities are presented in Section 5.4 to Section 5.7. We introduce in Section 5.3 important tools and properties to derive asymptotic solutions of differential equations. We have used these tools and properties to provide asymptotic solutions for the differential equation (3.1). These asymptotic solutions have been used to investigate the asymptotics of the eigenvalues of the above problems, the results can be found in Section 5.10 to Section 5.13.

## 5.2 Birkhoff regular boundary eigenvalue problems

### 5.2.1 Two-point boundary eigenvalue problems in $(L_2(a,b))^n$

Let $\Omega$ be a domain in $\mathbb{C}$, $-\infty < a < b < \infty$ and $n \in \mathbb{N}\backslash\{0\}$. Let

$$\begin{cases} T^D(\lambda)y = y' - A(\cdot,\lambda)y, \\ T^R(\lambda)y = W^a(\lambda)y(a) + W^b(\lambda)y(b), \end{cases} \tag{5.1}$$

for $\lambda \in \Omega$ and $y \in (W_1^2(a,b))^n$, where $A \in H(\Omega, M_n(L_\infty(a,b)))$ and $W^a,\ W^b \in H(\Omega, M_n(\mathbb{C}))$ with $\operatorname{rank}(W^a(\lambda), W^b(\lambda)) = n$ for all $\lambda \in \Omega$.

**Proposition 5.1.** *There are $\widetilde{A},\ \widetilde{B} \in H(\Omega, M_n(\mathbb{C}))$ such that the matrix*

$$\begin{pmatrix} W^a(\lambda)^\top & \widetilde{A}(\lambda) \\ W^b(\lambda)^\top & \widetilde{B}(\lambda) \end{pmatrix} \tag{5.2}$$

*is invertible for all $\lambda \in \Omega$.* See Proposition 3.5.1 [31, page 119].

**Lemma 5.2.** *Let $k > l \geq 1$, $x_j \in H(\Omega, \mathbb{C}^k)$ $(j = 1, \dots, l)$ and assume that the vectors $x_1(\lambda), \dots, x_l(\lambda)$ are linearly independent for all $\lambda \in \Omega$. Then there are $x_j \in H(\Omega, \mathbb{C}^k)$ $(j = l+1, \dots, k)$ such that $x_1(\lambda), \dots, x_k(\lambda)$ are linearly independent for all $\lambda \in \Omega$. In case $x_1, \dots, x_l$ are polynomials in $\lambda$, $x_{l+1}, \dots, x_k$ can be chosen to be polynomials.* See Lemma 3.5.2 [31, page 120].

**Definition 5.3.** [31, page 121]. We consider $T^{D^+} \in H(\mathbb{C}, L((W_1^2(a,b))^n, (L_2(a,b))^n))$ defined by

$$T^{D^+}(\lambda)y = -y' - A^\top(\cdot,\lambda)y \quad (\lambda \in \mathbb{C},\ y \in (W_1^2(a,b))^n. \tag{5.3}$$

The differential operator $T^{D^+}(\lambda)$ is called the formally adjoint of the differential operator $T^D(\lambda)$.

**Proposition 5.4.** *Set*

$$H := \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

*Then the Lagrange identity*

$$\langle T^D(\lambda)y, u\rangle - \langle y, T^{D^+}(\lambda)u\rangle = \begin{pmatrix} y(a) \\ y(b) \end{pmatrix}^\top H \begin{pmatrix} u(a) \\ u(b) \end{pmatrix}$$

*holds for all* $\lambda \in \Omega$*,* $y \in (W_1^2(a,b))^n$ *and* $u \in (W_1^2(a,b))^n$. See Proposition 3.5.4. [31, page 121].

**Definition 5.5.** [31, page 122] We define

$$\begin{pmatrix} \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \\ \widetilde{W}^a(\lambda) & \widetilde{W}^b(\lambda) \end{pmatrix} := \begin{pmatrix} W^a(\lambda)^\top & \widetilde{A}(\lambda) \\ W^b(\lambda)^\top & \widetilde{B}(\lambda) \end{pmatrix}^{-1} H, \tag{5.4}$$

where the matrix on the left-hand side is divided into $n \times n$ block matrices. In case $W^a(\lambda)$ and $W^b(\lambda)$ depend polynomially on $\lambda \in \mathbb{C}$, we can choose $\widetilde{A}(\lambda)$ and $\widetilde{B}(\lambda)$ to be polynomials in $\lambda$.

**Remark 5.6.** Since $\widetilde{A}$ and $\widetilde{B}$ are not uniquely determined, also the boundary matrix functions $\widetilde{W}^a$ and $\widetilde{W}^b$ are not uniquely determined. But the boundary condition

$$\widetilde{W}^a(\lambda)u(a) + \widetilde{W}^b(\lambda)u(b) = 0$$

are uniquely determined by the boundary conditions

$$W^a(\lambda)u(a) + W^b(\lambda)u(b) = 0.$$

See Remark 3.5.6. [31, page 123].

### 5.2.2 Definitions and basic properties

**Assumption 5.7.** [31, pages 130-133] Let $-\infty < a < b \leq \infty$ and $n \in \mathbb{N}\backslash\{0\}$. For sufficiently large complex numbers $\lambda$, say $|\lambda| \geq \gamma (> 0)$, we consider the boundary eigenvalue problem

$$y' - (\lambda A_1 + A_0 + \lambda^{-1}A^0(\cdot, \lambda))y = 0, \tag{5.5}$$

$$\sum_{j=0}^{\infty} \widetilde{W}^{(j)}(\lambda)y(a_j) + \int_a^b \widetilde{W}(x,\lambda)y(x)dx = 0, \tag{5.6}$$

where $y$ varies in $(W_1^2(a,b))^n$.

We assume that the coefficient matrices $A_0$ and $A_1$ belong to $M_n(L_2(a,b))$, that $A^0(\cdot,\lambda)$ belongs to $M_n(L_2(a,b))$ for $|\lambda| \geq \gamma$ and depends holomorphically on $\lambda$ there, and that

$$A^0(\cdot,\lambda) \text{ is bounded in } M_n(L_2(a,b)) \text{ as } \lambda \to \infty. \tag{5.7}$$

We suppose that $A_1$ is a diagonal matrix function, more precisely,

$$A_1 = \begin{pmatrix} A_0^1 & & & & \\ & A_1^1 & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \cdot \\ & & & & & A_l^1 \end{pmatrix},$$

where $l$ is a positive integer,

$$A_\nu^1 = r_\nu I_{n_\nu} \ (\nu = 0,\dots,l), \quad \sum_{\nu=0}^{l} n_\nu = n,$$

with $n_0 \in \mathbb{N}$ and $n_\nu \in \mathbb{N}\backslash\{0\}$ for $\nu = 1,\dots,l$. According to the block structure of $A_1$, we write $A_0 = (A_{0,\nu\mu})_{\nu,\mu=0}^{l}$.

For the diagonal elements of $A_1$ we assume:

I) $r_0 = 0$, and for $\nu,\mu = 0,\dots,l$ there are numbers $\phi_{\nu\mu} \in [0,2\pi)$ such that

$$(r_\nu - r_\mu)^{-1} \in L_\infty(a,b) \quad \text{if } \nu \neq \mu, \tag{5.8}$$

$$r_\nu(x) - r_\mu(x) = |r_\nu(x) - r_\mu(x)| e^{i\phi_{\nu\mu}} \quad \text{a.e. in } (a,b). \tag{5.9}$$

Note that $\mu = 0$ gives $r_\nu^{-1} \in L_\infty(a,b)$ for $\nu = 1,\dots,l$ and

$$r_\nu(x) = |r_\nu(x)| e^{i\phi_\nu} \quad \text{a.e in } (a,b) \quad (\nu = 1,\dots,l), \tag{5.10}$$

where $\phi_\nu := \phi_{\nu 0} = \phi_{0\nu} \pm \pi$ for $\nu = 1,\dots,l$.

If $n_0 = 0$, then we need conditions the conditions (5.8) and (5.9) only for $\nu,\mu \in \{1,\dots,l\}$. On the other hand, the conditions $r_\nu^{-1} \in L_\infty(a,b)$ for $\nu = 1,\dots,l$ and (5.10) are needed in any case. Hence it is no additional assumption if we take $\nu,\mu \in \{0,\dots,l\}$ in (5.8) and (5.9) also in the case $n_0 = 0$.

To give more explicit representation of condition I), we consider the following conditions (we remind that identities and inequalities of functions are understood to hold almost everywhere):

II) $r_0 = 0$, and there is a number $\alpha \in \mathbb{C}$ and for all $\nu = 0, \ldots, l$ real-valued functions $\widetilde{r}_\nu \in L_2(a,b)$ such that for all $\nu, \mu = 0, \ldots, l$

$$r_\nu = \alpha \widetilde{r}_\nu, \tag{5.11}$$

$$(r_\nu - r_\mu)^{-1} \in L_\infty(a,b) \text{ if } \nu \neq \mu, \tag{5.12}$$

$$\widetilde{r}_\nu - \widetilde{r}_\mu \text{ is positive or negative function if } \nu \neq \mu. \tag{5.13}$$

III) $r_0 = 0$, and there are a positive real-valued function $r \in L_2(a,b)$ such that $r^{-1} \in L_\infty(a,b)$ and $\alpha_\nu \in \mathbb{C}$ ($\nu = 0, \ldots, l$) such that $r_\nu = \alpha_\nu r$ and $\alpha_\nu \neq \alpha_\mu$ for $\nu, \mu = 0, \ldots, l$ and $\nu \neq \mu$.

**Remark 5.8.** [31, page 131] If $n_0 = 0$, then we need the conditions (5.8) and (5.9) only for $\nu, \mu \in \{1, \ldots, l\}$. On the other hand $r_\nu^{-1} \in L_\infty(a,b)$ for $\nu = 1, \ldots, l$ and (5.10) are needed in any case. Hence it is no additional assumption if we take $\nu, \mu \in \{0, \ldots, l\}$ in (5.8) and (5.9) also in the case $n_0 = 0$.

For the boundary conditions (5.6) we assume that the $a_j \in [a,b]$ for $j \in \mathbb{N}$, that $a_j \neq a_k$ if $j \neq k$, and that $a_0 = a$, $a_1 = b$. We suppose that the matrix function $\widetilde{W}(\cdot, \lambda)$ belongs to $M_n(L_1(a,b))$ for $|\lambda| \geq \gamma$ and there is $W_0 \in M_n(L_1(a,b))$ such that

$$\widetilde{W}(\cdot, \lambda) - W_0 = O(\lambda^{-1}) \quad \text{in } M_n(L_1(a,b)) \text{ as } \lambda \to \infty. \tag{5.14}$$

Finally we assume that the $\widetilde{W}_j(\lambda)$ are $n \times n$ matrices defined for $|\lambda| \geq \gamma$, and that there are $n \times n$ matrices $W_0^{(j)}$ such that the estimates

$$\sum_{j=0}^{\infty} |W_0^{(j)}| < \infty \tag{5.15}$$

and

$$\sum_{j=0}^{\infty} |\widetilde{W}^{(j)}(\lambda) - W_0^{(j)}| = O(\lambda^{-1}) \quad \text{as } \lambda \to \infty \tag{5.16}$$

hold.

**Proposition 5.9.** [31, page 133]. *There is a fundamental matrix function*

$$\widetilde{Y}(\cdot,\lambda) = (P^{[0]} + B_0(\cdot,\lambda))E(\cdot,\lambda) \tag{5.17}$$

*of the differential equation* (5.5) *having the following properties: the matrix function* $E(\cdot,\lambda)$ *belongs to* $M_n(W_1^2(a,b))$ *and*

$$E(x,\lambda) = \operatorname{diag}(E_0(x,\lambda), E_1(x,\lambda), \ldots, E_l(x,\lambda)) \tag{5.18}$$

*for* $x \in [a,b]$ *and* $\lambda \in \mathbb{C}$ *where*

$$R_\nu(x) = \int_a^x r_\nu \xi d\xi, \quad E_\nu(x,\lambda) = \exp(\lambda R_\nu(x)) I_{n_\nu}.$$

*The matrix function* $P^{[0]}$ *belongs to* $M_n(W_1^2(a,b))$ *and has block diagonal form according to the block structure of* $A_1$*, i.e.,*

$$P^{[0]} = \operatorname{diag}(P_{00}^{[0]}, P_{11}^{[0]}, \ldots, P_{ll}^{[0]}). \tag{5.19}$$

*The diagonal elements* $P_{\nu\nu}^{[0]}$ *are uniquely given as solutions of the initial value problems*

$$\begin{cases} P_{\nu\nu}^{[0]\prime} = A_{0,\nu\nu} P_{\nu\nu}^{[0]}, \\ P_{\nu\nu}^{[0]\prime}(a) = I_{n_\nu}, \end{cases} \tag{5.20}$$

*where the* $n_\nu \times n_\nu$ *matrix functions* $A_{0,\nu\nu}$ *are the block diagonal elements of* $A_0$*, the matrix function* $B_0(\cdot,\lambda)$ *belongs to* $M_n(W_1^2(a,b))$ *for* $|\lambda| \geq \gamma$ *and fulfils the estimates*

$$\begin{cases} B_0(\cdot,\lambda) = \{o(1)\}_\infty, \\ B_0(\cdot,\lambda) = \{O(\tau_p(\lambda))\}_\infty, \end{cases} \qquad \text{as } \lambda \to \infty, \tag{5.21}$$

*where*

$$\tau_p(\lambda) = \max_{\nu,\mu=0,\ \nu\neq\mu}^{l} (1 + |\Re(\lambda e^{i\phi_{\nu\mu}})|)^{-1+\frac{1}{p}}$$

*and* $\{o()\}_\infty$ *means that the estimate is uniform in* $x$*.*

**Notation 5.10.** [31, page 134]. For $\nu = 1, \ldots, l$ let $\phi_\nu$ be as defined in (5.10) and let $\lambda \in \mathbb{C}\backslash\{0\}$. We set

$$\delta_\nu(\lambda) := \begin{cases} 0 & \text{if } \Re(\lambda e^{i\phi_\nu}) < 0 \\ 1 & \text{if } \Re(\lambda e^{i\phi_\nu}) > 0 \\ 0 & \text{if } \Re(\lambda e^{i\phi_\nu}) = 0 \text{ and } \Im(\lambda e^{i\phi_\nu}) > 0, \\ 1 & \text{if } \Re(\lambda e^{i\phi_\nu}) = 0 \text{ and } \Im(\lambda e^{i\phi_\nu}) < 0. \end{cases} \tag{5.22}$$

Let $\delta_0(\lambda) = \delta_1(\lambda)$. We defined the block diagonal matrices

$$\begin{cases} \Delta(\lambda) := \operatorname{diag}(\delta_0 I_{n_0}, \ldots, \delta_l(\lambda) I_{n_l}), \\ \Delta_0 := \operatorname{diag}(0.I_{n_0}, I_{n_1}, \ldots, I_{n_l}), \end{cases} \tag{5.23}$$

which reduces to

$$\Delta(\lambda) = \operatorname{diag}(\delta_1(\lambda) I_{n_1}, \ldots, \delta_l(\lambda) I_{n_l}), \quad \Delta_0 := I_n, \tag{5.24}$$

if $n_0 = 0$. Finally we set

$$\widetilde{M_2} := \sum_{j=0}^{\infty} W_0^{(j)} P^{[0]}(a_j) + \int_a^b W_0(x)dx. \tag{5.25}$$

**Definition 5.11.** The boundary eigenvalue problem (5.5), (5.6) is called Birkhoff regular if

$$W_0^{(0)}(I_n - \Delta(\lambda))\Delta_0 + W_0^{(1)}\Delta(\lambda)\Delta_0 + \widetilde{M_2}(I_n - \Delta_0) \tag{5.26}$$

is invertible for $\lambda \in \mathbb{C}\backslash\{0\}$. See Definition 4.1.2. [31, page 134].

**Theorem 5.12.** *For $\nu = 1, \ldots, l$ we define*

$$\Lambda_\nu^1 := \begin{pmatrix} 0.I_{n_0} & & & & \\ & \delta_\nu^1 I_{n_1} & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \cdot & \\ & & & & & \delta_\nu^l I_{n_l} \end{pmatrix} \tag{5.27}$$

*and*

$$\Lambda_\nu^2 := \begin{pmatrix} 0.I_{n_0} & & & & \\ & (1-\delta_\nu^1) I_{n_1} & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \cdot & \\ & & & & & (1-\delta_\nu^l) I_{n_l} \end{pmatrix} \tag{5.28}$$

*where*

$$\delta_\nu^\mu := \begin{cases} 1 & \textit{if } \varphi \in [\varphi_\nu, \varphi_\nu + \pi) \mod (2\pi) \\ 1 & \textit{if } \varphi \notin [\varphi_\nu, \varphi_\nu + \pi) \mod (2\pi) \end{cases} \tag{5.29}$$

*for $\nu, \mu = 1, \ldots, l$. Here $\varphi_\mu \in Z \mod (2\pi)$ for a subset $Z$ of $\mathbb{R}$ means that there i a number $a \in Z$ such that $\varphi_\mu - a = 2\pi\mathbb{Z}$.*

*The boundary eigenvalue problem* (5.5), (5.6) *is Birkhoff regular if and only if the matrices*

$$W_0^{(0)}\Lambda_\nu^1 + W_0^{(1)}\Lambda_\nu^2 + \widetilde{M}_2(I_n - \Delta_0) \tag{5.30}$$

*and*

$$W_0^{(0)}\Lambda_\nu^2 + W_0^{(1)}\Lambda_\nu^1 + \widetilde{M}_2(I_n - \Delta_0) \tag{5.31}$$

*are invertible for all $\nu = 1, \ldots, l$, where $W_0^{(0)}$ and $W_0^{(1)}$ are uniquely determined by $\widetilde{W}^{(j)}(\lambda) - W_0^{(j)} = O(\lambda^{-1})$ as $\lambda \to \infty$ for $j = 0, 1$ and $\widetilde{M}_2$ is defined in* (5.25). See Theorem 4.1.3. [31, page 135].

**Proposition 5.13.** *Let $l = n$ and $\Delta(\lambda) = \operatorname{diag}(\delta_1(\lambda), \ldots, \delta_n(\lambda))$ as given by* (5.24). *We suppose that*

$$\phi_\nu = \frac{2\pi(\nu - 1)}{n} \quad (\nu = 1, \ldots, n),$$

i) *If $n$ is even, then the values of $\Delta$ are the diagonal matrices with $\frac{n}{2}$ consecutive ones and $\frac{n}{2}$ consecutive zeros in the diagonal in a cyclic arrangement.*

ii) *If $n$ is odd, then the values of $\Delta$ are the diagonal matrices with $\frac{n+1}{2}$ consecutive ones and $\frac{n-1}{2}$ consecutive zeros in the diagonal and the diagonal matrices with $\frac{n-1}{2}$ consecutive ones and $\frac{n+1}{2}$ consecutive zeros in the diagonal, each in a cyclic arrangement.*
See Proposition 4.1.7. [31, page 138].

### 5.2.3 Birkhoff regular problems

**Assumption 5.14.** [31, pages 204-205]. We consider the eigenvalue problem

$$y' - (\lambda A_1 + A_0)y = 0, \tag{5.32}$$

$$T^R(\lambda)y = \sum_{j=0}^{\infty} W^{(j)}(\lambda)y(a_j) + \int_a^b W(x, \lambda)y(x)dx = 0, \tag{5.33}$$

where $y \in (W_1^2(a,b))^n$, $-\infty < a < b < \infty$ and $n \in \mathbb{N}\backslash\{0\}$. We assume that there is an $n \times n$ matrix polynomial $C_2(\lambda)$ whose determinant is not identically zero such that the following properties hold:

There are matrix functions $W_0 \in M_n(L_1(a,b))$ such that

$$C_2^{-1}(\lambda)W(\cdot,\lambda) - W_0 = O(\lambda^{-1}) \quad \text{in } M_n(L_1(a,b)) \text{ as } \lambda \to \infty, \tag{5.34}$$

and there are $n \times n$ matrices $W_0^{(j)}$, $j \in \mathbb{N}$, such that the estimates

$$\sum_{j=0}^{\infty} |W_0^{(j)}| < \infty \tag{5.35}$$

and

$$\sum_{j=0}^{\infty} |C_2^{-1}(\lambda)W^{(j)}(\lambda) - W_0^{(j)}| = O(\lambda^{-1}) \quad \text{as } \lambda \to \infty \tag{5.36}$$

hold.

**Proposition 5.15.** [31, pages 205-206]. *There exists $\gamma > 0$ such that $C_2(\lambda)$ is invertible for $|\lambda| \geq \gamma$. For $|\lambda| \geq \gamma$ and $y \in (W_1^2(a,b))^n$ we set*

$$\widetilde{T}^R(\lambda)y = \sum_{j=0}^{\infty} C_2^{-1}(\lambda)W_j(\lambda)y(a_j) + \int_a^b C_2^{-1}(\lambda)W(x,\lambda)y(x)dx. \tag{5.37}$$

*Then $\widetilde{T}^R(\lambda)$ belongs to $\mathscr{L}(W_1^2(a,b))^n, \mathbb{C}^n)$ and depends holomorphically on $\lambda$ for $\lambda := \{\lambda \in \mathbb{C} : |\lambda| > \gamma\}$.*

**Remark 5.16.** [31, page 206] If $\nu$ is the maximum of the orders of the polynomials $W$ and $W^{(j)}$, then we can take $C_2(\lambda) = \lambda^\nu I_n$. In this case

$$C_2^{-1}(\lambda)W(\lambda) - W_0 = O(\lambda^{-1}) \tag{5.38}$$

and

$$C_2^{-1}(\lambda)W^{(j)}(\lambda) - W_0^{(j)} = O(\lambda^{-1}) \qquad (j \in \mathbb{N}) \tag{5.39}$$

hold for suitable $W_0$ and $W_0^{(j)}$.

**Definition 5.17.** The boundary eigenvalue problem (5.32), (5.33) is called Birkhoff regular if there is an $n \times n$ matrix polynomial $C_2(\lambda)$ fulfilling the assumption (5.34)–(5.36) such that the differential equation (5.32) with the boundary condition $\widetilde{T}^R(\lambda)y = 0$ is Birkhoff regular in the sense of Definition 5.11. See Defintion 5.2.1. [31, page 206].

**Assumption 5.18.** [31, page 281]. We consider the first order system $T^D(\lambda)y = 0$, where $T^D$ is given by (2.24), (2.25). We assume that there are a matrix function $C(\cdot,\lambda) \in M_n(W_1^2(a,b))$ depending polynomially on $\lambda$ and a positive real number $\gamma$ such that

$$C(\cdot,\lambda) \text{ is invertible in } M_n(W_1^2(a,b)) \text{ if } |\lambda| \geq \gamma \tag{5.40}$$

and such that the equation

$$C^{-1}(\cdot,\lambda)T^D(\lambda)C(\cdot,\lambda)y = y' - \widetilde{A}(\cdot,\lambda)y := \widetilde{T}^D(\lambda)y \tag{5.41}$$

holds for $|\lambda| \geq \gamma$ and $y \in (W_1^2(a,b))^n$, where

$$\widetilde{A}(\cdot,\lambda) = \lambda A_1 + A_0 + \lambda^{-1}A^0(\cdot,\lambda) \quad (|\lambda| \geq \gamma) \tag{5.42}$$

fulfils the assumptions made in Assumption 5.7.

**Proposition 5.19.** *Let $n_0 \in \{0,\dots,n-1\}$, $l := n - n_0$, and suppose that*

$$\pi(\cdot,\rho) = \rho^{n_0}\pi_l(\cdot,\rho),$$

*where*

$$\pi_l(\cdot,\rho) = \rho^l + \sum_{j=1}^{l}\rho^{l-j}\pi_{j,j} \tag{5.43}$$

*and $\pi_{n-i,j} \in L_2(a,b)$, $(i = 0,\dots,n-1,\ j = 0,\dots,n-i)$. Suppose that for all $x \in [a,b]$ the roots of $\pi_l(x,\rho) = 0$ are simple and nonzero and that there is $\kappa \in \mathbb{N}\backslash\{0\}$ such that $\pi_{1,1},\dots,\pi_{l,l} \in W_\kappa^2(a,b)$. Then there are $r_1,\dots,r_l \in W_\kappa^2(a,b)$ such that*

$$\pi_l(x,\rho) = \prod_{j=1}^{l}(\rho - r_j(x)) \tag{5.44}$$

*holds for all $x \in [a,b]$ and $\rho \in \mathbb{C}$. In addition, we have that $r_j^{-1} \in W_\kappa^2(a,b)$ for $j = 1,\dots,l$.* See Proposition 7.2.3 [31, pages 285-286].

**Theorem 5.20.** *Let $l \in \{1,\dots,n\}$ be such that $\pi_{l,l} \neq 0$ and $\pi_{i,i} = 0$ for $i = l+1,\dots,n$. Suppose that $\pi_{l,l} \in L_\infty(a,b)$. Then there is a matrix function*

$$C(x,\lambda) = \operatorname{diag}(\lambda^{\nu_1},\dots,\lambda^{\nu_n})C_1(x)$$

*with $\nu_1,\dots,\nu_n \in \mathbb{Z}$ and $C_1 \in M_n(W_1^2(a,b))$ such that $\widetilde{A}(\cdot,\lambda)$ given by (5.41) has the form (5.42), where*

$$A_1 = \operatorname{diag}(0,\dots,0,r_1,\dots,r_l)$$

*and $r_j^{-1} \in L_\infty(a,b)$ for $j = 1,\dots,l$, if and only the following conditions hold:*

i) $\pi_{l,l}^{-1} \in L_\infty(a,b)$;

ii)

$$p_i(\cdot,\lambda) = \sum_{j=0}^{l} \lambda^j \pi_{n-i,j} \qquad (i = 0,\dots,n_0-1);$$

iii) $\pi_{i,i} \in W_1^2(a,b)$ *for* $i = 1,\dots,l$ *or* $l = 1$ *and* $\frac{\pi_{n-i+1,1}}{\pi_{1,1}} \in W_1^2(a,b)$ *for* $i = 1,\dots,n-1$;

iv) *The zeros of* $\pi_l(x,\rho)$ *are simple and different from zero for all* $x \in [a,b]$, *where* $\pi_l$ *is defined in* (5.43).

**A** *If* i), ii), *and* iv) *hold and if* $l = 1$ *or* $\pi_{i,l} \in W_1^2(a,b)$ *for* $i = l+1,\dots,n$, *then we can choose* $\nu_1 = \cdots, \nu_{n_0+1} = 0$, $\nu_i = i - n_0 - 1$ $(i = n_0+2,\dots,n)$ *and*

$$C_1 = \begin{pmatrix} 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \\ -\frac{\pi_{n,l}}{\pi_{l,l}} & \cdots & -\frac{\pi_{l+1,l}}{\pi_{l,l}} & 1 & \cdots & 1 \\ & & & r_1 & \cdots & r_l \\ & 0 & & \vdots & & \vdots \\ & & & r_1^{l-1} & \cdots & r_l^{l-1} \end{pmatrix} \tag{5.45}$$

*where* $r_1,\dots,r_l \in W_1^2(a,b)$ *are the roots of* $\pi_l(\cdot,\rho) = 0$ *according to Proposition 5.19 if* $l > 1$.

**B** *If* i), ii) *and* iv) *hold and if* $\pi_{i,i} \in W_1^2(a,b)$ *for* $i = 1,\dots,l$, *then we can choose* $\nu_1 =$

*$\cdots = \nu_{n_0} = 0$, $\nu_i = i - n_0$ $(i = n_0+1, \ldots, n)$ and*

$$C_1 = \begin{pmatrix} 1 & & 0 & & & \\ & \cdot & & & 0 & \\ & & \cdot & & & \\ 0 & & 1 & r_1^{-1} & \cdots & r_l^{-1} \\ & & & & & \\ & & & 1 & \cdots & 1 \\ & & & r_1 & \cdots & r_l \\ & 0 & & \vdots & & \vdots \\ & & & r_1^{l-1} & \cdots & r_l^{l-1} \end{pmatrix} \tag{5.46}$$

*where $r_1, \ldots, r_l \in W_1^2(a,b)$ are the roots of $\pi_l(\cdot, \rho) = 0$ according to Proposition 5.19. Note that in case $l = n$ the matrix $C_1$ is just the lower right block.*

See Theorem 7.2.4. [31, pages 288-289].

**Definition 5.21.** Together with the boundary conditions (2.39) and a function $C(x,\lambda)$ satisfying (5.40)–(5.42) we consider the matrix functions

$$\begin{cases} W^{(j)}(\lambda) := (w_{ki}^{(j)}(\lambda))_{k,i=1}^n C(a_j, \lambda), \\ W(x,\lambda) := (w_{ki},(\lambda))_{k,i=1}^n C(x,\lambda) \end{cases} \tag{5.47}$$

and set

$$\widehat{T}^R(\lambda)y := \sum_{j=0}^{\infty} W^{(j)}(\lambda) y(a_j) + \int_a^b W(x,\lambda)y(x)dx \qquad (y \in (W_1^2(a,b))^n). \tag{5.48}$$

The boundary eigenvalue problem (2.38), (2.39) is called Birkhoff regular if $\pi_{nn} \neq 0$ and if there are matrix functions $C(\cdot,\lambda)$ satisfying (5.40)–(5.42) and $C_2(\lambda)$ satisfying (5.34)–(5.36) so that the associated boundary eigenvalue problem $\widetilde{T}^D(\lambda)y = 0$, $C_2(\lambda)^{-1}\widehat{T}^R(\lambda)y = 0$ is Birkhoff regular in the sense of Definition 5.11. See Definiton 7.3.1. [31, page 295].

## 5.3 An asymptotic fundamental system for $K\eta = \lambda^l H\eta$

Let $-\infty < a < b < \infty$ and $n \in \mathbb{N}$, $n \geq 2$. We consider the differential equation

$$K\eta = \lambda^l H\eta \quad (\eta \in W_n^1(a,b)), \tag{5.49}$$

where

$$K\eta = \eta^{(n)} + \sum_{i=0}^{n-1} k_i \eta^{(i)}, \tag{5.50}$$

$$H\eta = \sum_{i=0}^{n_0} h_i \eta^{(i)}, \tag{5.51}$$

with $0 \leq n_0 \leq n-1$ and $k_i, h_i \in W_i^2(a,b)$. We shall always assume that $h_{n_0} > 0$ and $h_{n_0}^{-1} \in L_\infty(a,b)$. We infer that $h_{n_0}^{-1} \in W_1^2(a,b)$ if $n_0 > 0$. If $n_0 = 0$, we suppose that $h_{n_0}^{-1} \in W_1^2(a,b)$. We associate the differential operator

$$L^D(\lambda)\eta := K\eta - \lambda^l H\eta \quad (\eta \in W_n^1(a,b)) \tag{5.52}$$

with the differential equation (5.49). Together with the differential equation (5.49) we consider the two point boundary conditions

$$L^R(\lambda)\eta := \left( \sum_{i=0}^{n-1} w_{ki}^{(0)}(\lambda)\eta^{(i-1)}(a) + \sum_{i=0}^{n-1} w_{ki}^{(1)}(\lambda)\eta^{(i-1)}(b) \right)_{k=1}^{n} = 0, \tag{5.53}$$

where $w_{ki}^{(j)}$ are polynomials.

**Definition 5.22.** [31, page 322]. We define the boundary matrices

$$W^{(j)}(\lambda) = (w_{ki}^{(j)}(\lambda))_{k,i=1}^{n} \quad (j = 0, 1). \tag{5.54}$$

**Theorem 5.23.** *Suppose that $h_{n_0} = 1$, set $l = n - n_0$ and let $k \in \mathbb{N}$. Assume that $k \geq \max\{l, n_0 - 1\}$ if $n_0 > 0$. Suppose that*

*$\alpha$) $k_j \in L_2(a,b)$ for $j = 0, \ldots, n-1-k$ and $k_{n-1-j} \in W_{k-j}^2(a,b)$ for $j = 0, \ldots, \min\{k-1, n-1\}$ if $n_0 = 0$,*

*$\beta$) $h_0, \ldots, h_{n_0-1} \in W_k^2(a,b)$, $k_0, \ldots, k_{n_0-1} \in W_{k-l}^2(a,b)$ and $k_{n-1-j} \in W_{k-j}^2(a,b)$ for $j = 0, \ldots, l-1$ if $n_0 > 0$. Let $\{\pi_1, \ldots, \pi_{n_0}\} \subset W_{k+n_0}^2(a,b)$ be a fundamental system of $H\eta = 0$.*

*For sufficiently large* $\lambda$ *the differential equation* $K\eta = \lambda^l H\eta$ *has a fundamental system* $\{\eta_1(\cdot,\lambda),\dots,\eta_n(\cdot,\lambda)\}$ *with the following properties:*

i) *There are functions* $\pi_{\nu r} \in W_{k+n_0-lr}(a,b)$ $(1 \le \nu \le n_0,\ 1 \le r \le [\frac{k}{l}])$ *such that*

$$\eta_\nu^{(\mu)}(\cdot,\lambda) = \pi_\nu^{(\mu)} + \sum_{r=1}^{[\frac{k}{l}]} \lambda^{-lr}\pi_{\nu r}^{(\mu)} + \{o(\lambda^{-k})\}_\infty$$
$$(\nu = 1,\dots,n_0;\ \mu = 0,\dots,n_0-1), \tag{5.55}$$

$$\eta_\nu^{(\mu)}(\cdot,\lambda) = \pi_\nu^{(\mu)} + \sum_{r=1}^{[\frac{k-\mu+n_0-1}{l}]} \lambda^{-lr}\pi_{\nu r}^{(\mu)} + \{o(\lambda^{-k+\mu-n_0+1})\}_\infty$$
$$(\nu = 1,\dots,n_0;\ \mu = 0,\dots,n_0-1). \tag{5.56}$$

ii) *Set* $\widetilde{k} := \min\{k, k+1-n_0\}$. *Let* $\omega_j = \exp\{\frac{2\pi i(j-1)}{l}\}$ $(j = 1,\dots,l)$. *There are functions* $\varphi_r \in W^2_{k+1-r}(a,b)$, $r = 0,\dots,\widetilde{k}$, *such that* $\varphi_0$ *is solution of the initial value problem*

$$\varphi_0' - \frac{1}{l}(h_{n_0-1} - k_{n-1})\varphi_0 = 0, \quad \pi_0(a) = 1, \tag{5.57}$$

*and*

$$\eta_\nu^{(\mu)}(x,\lambda) = \left[\frac{d^\mu}{dx^\mu}\right]\left[\sum_{r=0}^{\widetilde{k}}(\lambda\omega_{\nu-n_0})^{-r}\varphi_r(x)e^{\lambda\omega_{\nu-n_0}(x-a)}\right] + \{o(\lambda^{-\widetilde{k}+\mu})\}_\infty e^{\lambda\omega_{\nu-n_0}(x-a)}$$
$$(\nu = n_0+1,\dots,n;\ \mu = 0,\dots,n_0-1), \tag{5.58}$$

*where* $\left[\frac{d^\mu}{dx^\mu}\right]$ *means that we omit those terms of Leibniz expansion which contain a function* $\varphi_r^{(j)}$ *with* $j > \widetilde{k} - r$. See Theorem 8.2.1. [31, page 326].

**Proposition 5.24.** [31, pages 327-332] *With the assumptions of Theorem 5.23 we denote the i-th vectors in* $\mathbb{C}^n$, $\mathbb{C}^{n_0}$, $\mathbb{C}^l$ *by* $e_i$, $\epsilon_i$, $\varepsilon_i$. *For* $i \in \mathbb{Z}\backslash\{1,\dots,n\}$ *or* $i \in \mathbb{Z}\backslash\{1,\dots,n_0\}$ *or*

*$i \in \mathbb{Z}\backslash\{1,\dots,l\}$, we set $e_i := 0$, $\epsilon_i := 0$, $\varepsilon_i := 0$, respectively. We set*

$$a_{11}^\top := (h_0,\dots,h_{n_0-1}) \tag{5.59}$$

$$a_{12}^\top := -(k_0,\dots,k_{n_0-1}) \tag{5.60}$$

$$J_r := \begin{pmatrix} 0 & 1 & & & & \\ & 0 & \cdot & & 0 & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & 0 & & & \cdot & 1 \\ & & & & & 0 \end{pmatrix} \in M_r(\mathbb{C}) \tag{5.61}$$

$$\varepsilon^\top := \sum_{i=1}^{l} \varepsilon_i^\top = (1,\dots,1) \in \mathbb{C}^l, \tag{5.62}$$

$$\Omega_l := \operatorname{diag}(\omega_1,\dots,\omega_l), \tag{5.63}$$

$$\Xi_r(\lambda) := \operatorname{diag}(1,\lambda\dots,\lambda^{r-1}) \in M_r(\mathbb{C}), \tag{5.64}$$

$$V := \sum_{i=1}^{l} \varepsilon_i \varepsilon^\top \Omega_l^{i-1} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & 1 \\ \omega_1 & \cdot & \cdot & \cdot & \omega_l \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \omega_1^{l-1} & \cdot & \cdot & \cdot & \omega_l^{l-1} \end{pmatrix}. \tag{5.65}$$

*For $\lambda$ sufficiently large, let*

$$E(x,\lambda) = \operatorname{diag}(1,\dots,1,e^{\lambda\omega_1(x-a)},\dots,e^{\lambda\omega_l(x-a)}), \tag{5.66}$$

$$M_1(\cdot,\lambda) = \begin{pmatrix} \widetilde{Q}_{11}(\cdot,\lambda) + \epsilon_{n_0}\varepsilon^\top\Omega_l^{-1}\widetilde{Q}_{21}(\cdot,\lambda) & \lambda^{n_0-1}\widetilde{Q}_{12}(\cdot,\lambda)\Omega_l^{n_0} + \lambda^{n_0-1}\epsilon_{n_0}\varepsilon^\top\Omega_l^{-1}\widetilde{Q}_{22}(\cdot,\lambda)\Omega_l^{n_0} \\ \lambda\Xi_l(\lambda)V\widetilde{Q}_{21}(\cdot,\lambda) & \lambda^{n_0}\Xi_l V\widetilde{Q}_{22}(\cdot,\lambda)\Omega_l^{n_0} \end{pmatrix}, \tag{5.67}$$

$$M_2(\cdot,\lambda) = \begin{pmatrix} D & 0 \\ 0 & I_l \end{pmatrix}, \tag{5.68}$$

*where the matrix $D$ is a $n_0 \times n_0$ that can be chosen such that*

$$(\pi_1,\dots,\pi_{n_0}) = \epsilon_1^\top Q^{[0]} D \tag{5.69}$$

$$Q_{11}^{[0]\prime} - (J_{n_0} - \epsilon_{n_0} a_{11}^\top) Q_{11}^{[0]} = 0, \quad Q_{11}^{[0]}(a) = I_{n_0}, \tag{5.70}$$

$$Q_{22}^{[0]\prime} - \frac{1}{l}(h_{n_0-1} - k_{n-1}) Q_{22}^{[0]} = 0, \quad Q_{22}^{[0]}(a) = I_l, \tag{5.71}$$

$$Q_{12}^{[0]} = 0 \qquad Q_{21}^{[0]} = 0, \tag{5.72}$$

$$\left.\begin{array}{l} Q_{11}^{[r]\prime} - (J_{n_0} - \epsilon_{n_0} a_{11}^\top) Q_{11}^{[r]} = (\epsilon_{n_0-1} - (h_{n_0-1} - k_{n-1})\epsilon_{n_0}) \varepsilon^\top \Omega_l^{-1} Q_{21}^{[r]} \\ + \sum\limits_{j=1}^{l} k_{n-1-j} \epsilon_{n_0} \varepsilon^\top \Omega_l^{-1-j} Q_{21}^{[r-j]} - \epsilon_{n_0} a_{12}^\top Q_{11}^{[r-l]} \quad (r = 1, \ldots, k), \end{array}\right\} \tag{5.73}$$

$$\left.\begin{array}{l} Q_{21}^{[r]} = \Omega_l^{-1} Q_{21}^{[r-1]\prime} - \frac{1}{l} \varepsilon a_{11}^\top Q_{11}^{[r-1]} - \frac{1}{l}(h_{n_0-1} - k_{n-1}) \varepsilon \varepsilon^\top \Omega_l^{-1} Q_{21}^{[r-1]} \\ + \frac{1}{l} \sum\limits_{j=1}^{l} k_{n-1-j} \varepsilon \varepsilon^\top \Omega_l^{-1-j} Q_{21}^{[r-1-j]} - \frac{1}{l} \varepsilon a_{12}^\top Q_{11}^{[r-l-1]} \quad (r = 1, \ldots, k), \end{array}\right\} \tag{5.74}$$

$$\left.\begin{array}{l} Q_{12}^{[r]} = -Q_{12}^{[r-1]\prime} \Omega_l^{-1} + (J_{n_0} - \epsilon_{n_0} a_{11}^\top) Q_{12}^{[r-1]} \Omega_l^{-1} \\ + (\epsilon_{n_0-1} - (h_{n_0-1} - k_{n-1})\epsilon_{n_0}) \varepsilon^\top \Omega_l^{-1} Q_{22}^{[r-1]} \Omega_l^{-1} \\ + \sum\limits_{j=1}^{l} k_{n-1-j} \epsilon_{n_0} \varepsilon^\top \Omega_l^{-1-j} Q_{22}^{[r-1-j]} \Omega_l^{-1} - \epsilon_{n_0} a_{12}^\top Q_{12}^{[r-1-l]} \Omega_l^{-1} \quad (r = 1, \ldots, k), \end{array}\right\} \tag{5.75}$$

$$\left.\begin{array}{l} \Omega_l Q_{22}^{[r]} - Q_{22}^{[r]} \Omega_l = Q_{22}^{[r-1]\prime} - \frac{1}{l} \Omega_l \varepsilon a_{11}^\top Q_{12}^{[r-1]} - \frac{1}{l}(h_{n_0-1} - k_{n-1}) \Omega_l \varepsilon \varepsilon^\top \Omega_l^{-1} Q_{22}^{[r-1]} \\ + \frac{1}{l} \sum\limits_{j=1}^{l} k_{n-1-j} \Omega_l \varepsilon \varepsilon^\top \Omega_l^{-1-j} Q_{22}^{[r-1-j]} - \frac{1}{l} \Omega_l \varepsilon a_{12}^\top Q_{12}^{[r-1-l]} \quad (r = 1, \ldots, k) \end{array}\right\} \tag{5.76}$$

$$\left.\begin{array}{l} 0 = \varepsilon_\nu^\top \Big( Q_{22}^{[k]\prime} - \frac{1}{l} \Omega_l \varepsilon a_{11}^\top Q_{12}^{[k]} - \frac{1}{l}(h_{n_0-1} - k_{n-1}) \Omega_l \varepsilon \varepsilon^\top \Omega_l^{-1} Q_{22}^{[k]} \\ + \frac{1}{l} \sum\limits_{j=1}^{l} k_{n-1-j} \Omega_l \varepsilon \varepsilon^\top \Omega_l^{-1-j} Q_{22}^{[k-j]} - \frac{1}{l} \Omega_l \varepsilon a_{12}^\top Q_{12}^{[k-l]} \Big) \varepsilon_\nu \quad (\nu = 1, \ldots, l) \end{array}\right\} \tag{5.77}$$

*and*

$$\widetilde{Q}_{ij}(\cdot, \lambda) = \sum_{r=0}^{k} \lambda^{-r} Q_{ij}^{[r]} + o\{(\lambda^{-k})\}_\infty \qquad (i, j = 1, 2). \tag{5.78}$$

*We set* $\eta_\nu := \eta_{0,\nu}$. *Then*

$$(\eta_{\mu-1,\nu}(\cdot, \lambda))_{\mu,\nu=1}^{n} := \left(\eta_\nu^{\mu-1}(\cdot, \lambda)\right)_{\mu,\nu=1}^{n} := Y(\cdot, \lambda) := M_1(\cdot, \lambda) M_2(\cdot, \lambda) E(\cdot, \lambda) \tag{5.79}$$

*is a fundamental matrix of* $T^D(\lambda) = 0$ *if* $\lambda$ *sufficiently large, where* $T^D(\lambda) = \begin{pmatrix} 0 \\ \vdots \\ L^D(\lambda) \end{pmatrix}$, *and* $\{\eta_1(\cdot, \lambda), \ldots, \eta_n(\cdot, \lambda)\}$ *is a fundamental system of the differential equation* $K\eta - \lambda^l H \eta = 0$.

**Proposition 5.25.** *Let* $n_0 \geq 2$. *For* $r = 0, \ldots, n_0 - 2$ *and* $i = 1, \ldots, n_0 - r - 1$ *we have* $\epsilon_i^\top Q_{12}^{[r]} = 0$. *For* $r = 1, \ldots, n_0 - 1$ *we have* $\epsilon_{n_0-r}^\top Q_{12}^{[r]} \varepsilon_1 = \varepsilon_1^\top Q_{22}^{[0]} \varepsilon_1 = 0$.

See Proposition 8.2.3. [31, page 335].

**Remark 5.26.** Let $n_0 \geq 2$. Then Proposition 5.25 yields $\epsilon_1^\top Q_{12}^{[r]} = 0$ for $r = 0, \ldots, n_0 - 2$.

Hence,

**Corollary 5.27.** [31, page 335] *By* (5.78), *and* (5.79), *there are* $\varphi_{\nu r} \in W^2_{k+2-n_0-r}(a,b)$ *for* $\nu = n_0 + 1, \ldots, n$ *and* $r = 0, \ldots, k+1-n_0$ *such that, for* $\nu = n_0 + 1, \ldots, n$

$$\eta_\nu(x,\lambda) = \left\{ \sum_{r=0}^{k+1-n_0} \lambda^{-r} \varphi_{\nu r}(x) + \{o(\lambda^{-k-1+n_0})\}_\infty \right\} e^{\lambda \omega_{\nu - n_0} x}. \tag{5.80}$$

*The equation* (5.80) *also holds for* $n_0 = 1$ *and that for* $n_0 = 0$ *and* $\nu = 1, \ldots, n$,

$$\eta_\nu(x,\lambda) = \left\{ \sum_{r=0}^{k} \lambda^{-r} \varphi_{\nu r}(x) + \{o(\lambda^{-k})\}_\infty \right\} e^{\lambda \omega_{\nu - n_0} x}, \tag{5.81}$$

*where* $\varphi_{\nu r} \in W^2_{k+1-r}(a,b)$.

**Proposition 5.28.** [31, page 336] *With the assumptions of Theorem 5.23 and notations of Proposition 5.24, if* $n_0 = 0$, $1 \leq \nu \leq n$, *and* $0 \leq r \leq k$, *where* $k$ *is as defined in Theorem 5.23, then*

$$\varphi_{\nu r} = \varepsilon_1^\top V Q_{22}^{[r]} \varepsilon_\nu, \tag{5.82}$$

$$\varphi_0 = \varphi_{10} = \varepsilon_1^\top V Q_{22}^{[r]} \varepsilon_1. \tag{5.83}$$

**Note 5.29.** In the following sections, we investigate the Birkhoff regularity of eigenvalue problem (3.1)–(3.2) where $g \neq 0$, and different cases of the boundary conditions $B_1(y) = y^{[p_1]} = 0$, $B_2(y) = y^{[p_2]} = 0$, where $p_1 + p_2 \neq 3$, $B_3(y) = y^{[p_3]}(a) + i\varepsilon_3 \alpha \lambda y^{[q_3]}(a) = 0$ and $B_4(y) = y^{[p_4]}(a) + i\varepsilon_4 \alpha \lambda y^{[q_4]}(a) = 0$, where for $j = 3, 4$, $p_j + q_j = 3$, $\varepsilon_j = 1$ if $q_j$ is odd and $\varepsilon_j = -1$ if $q_j$ is even.

## 5.4 Birkhoff regularity for $B_1(y) = y(0) = 0$ and $B_2(y) = y''(0) = 0$

Recall that $\lambda = \mu^2$. Then $p_i(\cdot,\mu) = \sum\limits_{i=0}^{3} \mu^j \pi_{3-i,j}$, $(i = 0,1,2,3,\ j = 0,1,2,3)$, see (2.35). Hence it follows from (2.35) and (2.38) that $\pi_{1,1} = \pi_{2,2} = \pi_{3,3} = 0$, while $\pi_{4,4} = -1$. Thus the characteristic function of (3.1) is $\pi(\rho) = \rho^4 - 1$, see (2.36), its zeros are $i^{k-1}$, $k = 1,2,3,4$. According to Proposition 5.13 the matrices $\Delta$ of the eigenvalue problem (3.1)–(3.2) are the following four $4 \times 4$ diagonal matrices with two consecutive ones and two consecutive zeros in the diagonal in a cycle arrangement:

$$\begin{cases} \Delta_1 = \operatorname{diag}(1,1,0,0) \\ \Delta_2 = \operatorname{diag}(0,1,1,0) \\ \Delta_3 = \operatorname{diag}(0,0,1,1) \\ \Delta_4 = \operatorname{diag}(1,0,0,1) \end{cases}. \tag{5.84}$$

Since $n_0 = 0$, then it follows from Proposition 5.19 that $l = 4$. Then the matrix $C_1$ defined in (5.45) is reduced to $(i^{(k-1)(l-1)})_{k,l=1}^4$. It follows from Theorem 5.20.A that $\nu_1 = 0$, $\nu_2 = 1$, $\nu_3 = 2$ and $\nu_4 = 3$. Hence we can choose

$$\begin{aligned} C(x,\mu) &= \operatorname{diag}\left(1,\mu,\mu^2,\mu^3\right)\left(i^{(k-1)(l-1)}\right)_{k,l=1}^4 \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu^2 & 0 \\ 0 & 0 & 0 & \mu^3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mu & i\mu & -\mu & -i\mu \\ \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ \mu^3 & -i\mu^3 & -\mu^3 & i\mu^3 \end{pmatrix}. \end{aligned} \tag{5.85}$$

The matrix $C(x,\mu)$ is also considered in the sections Section 5.5 to Section 5.7.

The following boundary matrix is associated to the problems of this section

$$W^{(0)}(\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} C(0,\mu) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.86}$$

see (5.47).

### 5.4.1 The boundary conditions $B_3y = y''(a) + i\alpha\lambda y'(a) = 0$ and $B_4y = y^{(3)}(a) - i\alpha\lambda y(a) = 0$

According to (5.47) the second boundary matrix associated of this problem is given by

$$W^{(1)}(\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i\alpha\mu^2 & 1 & 0 \\ -i\alpha\mu^2 & 0 & 0 & 1 \end{pmatrix} C(a,\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}, \tag{5.87}$$

where $\gamma_j = i^j\alpha\mu^3 + (-1)^{j-1}\mu^2$ and $\beta_j = -i\alpha\mu^2 + (-i)^{j-1}\mu^3$. Choosing $C_2(\mu) = \mathrm{diag}(1, \mu^2, \mu^3, \mu^3)$, then

$$\begin{aligned} C_2(\mu)^{-1}W^{(0)}(\mu) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu^{-2} & 0 & 0 \\ 0 & 0 & \mu^{-3} & 0 \\ 0 & 0 & 0 & \mu^{-3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = W_0^{(0)} \end{aligned} \tag{5.88}$$

and

$$\begin{aligned} C_2(\mu)^{-1}W^{(1)}(\mu) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu^{-2} & 0 & 0 \\ 0 & 0 & \mu^{-3} & 0 \\ 0 & 0 & 0 & \mu^{-3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \\ \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\alpha & -\alpha & -i\alpha & \alpha \\ 1 & -i & -1 & i \end{pmatrix} + O(\mu^{-1}) \\ &= W_0^{(1)} + O(\mu^{-1}), \end{aligned} \tag{5.89}$$

where $\delta_j = i^j\alpha + (-1)^{j-1}\mu^{-1}$, $\theta_j = (-i)^{j-1} - i\alpha\mu^{-1}$ and

$$W_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\alpha & -\alpha & -i\alpha & \alpha \\ 1 & -i & -1 & i \end{pmatrix}. \tag{5.90}$$

Thus $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39). It is easy to check that

$$|W_0^{(0)}| = 2, \tag{5.91}$$

while

$$|W_0^{(1)}| = 1 + \alpha. \tag{5.92}$$

Thus $|W_0^{(0)}| + |W_0^{(1)}| = 3 + \alpha < \infty$. On the other hand it is clear that

$$|C_2(\mu)^{-1}W^{(0)}(\mu) - W_0^{(0)}| = 0, \tag{5.93}$$

while

$$|C_2(\mu)^{-1}W^{(1)}(\mu) - W_0^{(1)}| = O(\mu^{-1}). \tag{5.94}$$

Hence

$$|C_2(\mu)^{-1}W^{(0)}(\mu) - W_0^{(0)}| + |C_2(\mu)^{-1}W^{(1)}(\mu) - W_0^{(1)}| = O(\mu^{-1}). \tag{5.95}$$

Whence (5.35) and (5.36) hold.

### 5.4.2 The boundary conditions $B_3y = y''(a) + i\alpha\lambda y'(a) = 0$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a) = 0$

The associated boundary matrices are the matrix $W^{(0)}(\mu)$ given in (5.86) and the matrix

$$W^{(1)}(\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i\alpha\mu^2 & 1 & 0 \\ 1 & 0 & 0 & i\alpha\mu^2 \end{pmatrix} C(a,\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}, \tag{5.96}$$

where $\gamma_j = i^j\alpha\mu^3+(-1)^{j-1}\mu^2$ and $\beta_j = 1+(-1)^{j-1}i^j\alpha\mu^5$. Choosing $C_2(\mu) = \mathrm{diag}(1,\mu^2,\mu^3,\mu^5)$, then

$$\begin{aligned}
C_2(\mu)^{-1}W^{(1)}(\mu) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu^{-2} & 0 & 0 \\ 0 & 0 & \mu^{-3} & 0 \\ 0 & 0 & 0 & \mu^{-5} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \\ \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\alpha & -\alpha & -i\alpha & \alpha \\ i\alpha & \alpha & -i\alpha & -\alpha \end{pmatrix} + O(\mu^{-1}) \\
&= W_0^{(1)} + O(\mu^{-1}),
\end{aligned} \tag{5.97}$$

where $\delta_j = i^j\alpha + (-1)^{j-1}\mu^{-1}$, $\theta_j = (-1)^{j-1}i^j\alpha + \mu^{-5}$ and

$$W_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\alpha & -\alpha & -i\alpha & \alpha \\ i\alpha & \alpha & -i\alpha & -\alpha \end{pmatrix}. \tag{5.98}$$

As the matrix $W^{(0)}(\mu)$ defined in (5.86) is associated to the eigenvalue problem defined in this subsection and the first two diagonal entries of the matrix $C_2(\mu)$ are the same as for the matrix $C_2(\mu)$ defined in Subsection 5.4.1, then $C_2(\mu)^{-1}W^{(0)}(\mu) = W_0^{(0)}$, where $W_0^{(0)}$ is the matrix obtained in (5.88). Thus $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39).

### 5.4.3 The boundary conditions $B_3y = y'(a) - i\alpha\lambda y''(a) = 0$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a) = 0$

The boundary matrices associated to this problem are the matrix $W^{(0)}(\mu)$ given in (5.86) and the matrix

$$W^{(1)}(\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -i\alpha\mu^2 & 0 \\ 1 & 0 & 0 & i\alpha\mu^2 \end{pmatrix} C(a,\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}, \tag{5.99}$$

where $\gamma_j = i^{j-1}\mu + (-1)^j i\alpha\mu^4$ and $\beta_j = 1 + (-1)^{j-1} i^j \alpha\mu^5$. Choosing $C_2(\mu) = \operatorname{diag}(1, \mu^2, \mu^4, \mu^5)$, then

$$\begin{aligned} C_2(\mu)^{-1}W^{(1)}(\mu) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu^{-2} & 0 & 0 \\ 0 & 0 & \mu^{-4} & 0 \\ 0 & 0 & 0 & \mu^{-5} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \\ \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\alpha & -i\alpha & i\alpha & -i\alpha \\ i\alpha & \alpha & -i\alpha & -\alpha \end{pmatrix} + O(\mu^{-1}) \\ &= W_0^{(1)} + O(\mu^{-1}), \end{aligned} \tag{5.100}$$

where $\delta_j = (-1)^j i\alpha + i^{j-1}\mu^{-3}$, $\theta_j = (-1)^{j-1} i^j \alpha + \mu^{-5}$ and

$$W_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\alpha & -i\alpha & i\alpha & -i\alpha \\ i\alpha & \alpha & -i\alpha & -\alpha \end{pmatrix}. \tag{5.101}$$

On the other hand the first two diagonal entries of the matrix $C_2(\mu)$ are the same as the first two diagonal entries of the matrix $C_2(\mu)$ defined in Subsection 5.4.1, then $C_2(\mu)^{-1}W^{(0)}(\mu) =$

$W_0^{(0)}$, where $W_0^{(0)}$ is the matrix given in (5.88). Thus $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39).

### 5.4.4 The boundary conditions $B_3 y = y'(a) - i\alpha\lambda y''(a) = 0$ and $B_4 y = y^{(3)}(a) - i\alpha\lambda y(a) = 0$

The boundary matrices of this problem are the matrix $W^{(0)}(\mu)$ given in (5.86) and the matrix

$$W^{(1)}(\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -i\alpha\mu^2 & 0 \\ -i\alpha\mu^2 & 0 & 0 & 1 \end{pmatrix} C(a,\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}, \tag{5.102}$$

where $\gamma_j = i^{j-1}\mu + (-1)^j i\alpha\mu^4$ and $\beta_j = -i\alpha\mu^2 + (-i)^{j-1}\mu^3$. Choosing $C_2(\mu) = \mathrm{diag}(1, \mu^2, \mu^4, \mu^3)$, then

$$\begin{aligned} C_2(\mu)^{-1}W^{(1)}(\mu) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu^{-2} & 0 & 0 \\ 0 & 0 & \mu^{-4} & 0 \\ 0 & 0 & 0 & \mu^{-3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \\ \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\alpha & -i\alpha & i\alpha & -i\alpha \\ 1 & -i & -1 & i \end{pmatrix} \\ &= W_0^{(1)} + O(\mu^{-1}), \end{aligned} \tag{5.103}$$

where $\delta_j = (-1)^j i\alpha + i^{j-1}\mu^{-3}$, $\theta_j = (-i)^{j-1} - i\alpha\mu^{-1}$ and

$$W_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\alpha & -i\alpha & i\alpha & -i\alpha \\ 1 & -i & -1 & i \end{pmatrix}. \tag{5.104}$$

Since the first two diagonal entries of the matrix $C_2(\mu)$ are identical to the first two diagonal entries of the matrix $C_2(\mu)$ of Subsection 5.4.4, then $C_2(\mu)^{-1}W^{(0)}(\mu) = W_0^{(0)}$, where $W_0^{(0)}$ is the matrix obtained in (5.88). It follows that $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39).

## 5.5 Birkhoff regularity for $B_1(y) = y(0) = 0$ and $B_2(y) = y'(0) = 0$

The following boundary matrix is associated to all of the eigenvalue problems of this sections

$$W^{(0)}(\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} C(0,\mu) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mu & i\mu & -\mu & -i\mu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.105}$$

### 5.5.1 The boundary conditions $B_3y = y''(a) + i\alpha\lambda y'(a) = 0$ and $B_4y = y^{(3)}(a) - i\alpha\lambda y(a) = 0$

The second boundary matrix associated to this problem is, according to (5.47), given by the matrix $W^{(1)}(\mu)$ defined in (5.87). Choosing $C_2(\mu) = \mathrm{diag}(1, \mu, \mu^3, \mu^3)$, then

$$\begin{aligned} C_2(\mu)^{-1}W^{(0)}(\mu) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu^{-1} & 0 & 0 \\ 0 & 0 & \mu^{-3} & 0 \\ 0 & 0 & 0 & \mu^{-3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mu & i\mu & -\mu & -i\mu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = W_0^{(0)} \end{aligned} \tag{5.106}$$

and $C_2(\mu)^{-1}W^{(1)}$ yields the matrix given in (5.90), as the last two diagonal entries of the matrix $C_2(\mu)$ are identical to the last two diagonal entries of the matrix $C_2(\mu)$ of Subsection

5.4.1. Thus $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39).

### 5.5.2 The boundary conditions $B_3y = y''(a) + i\alpha\lambda y'(a) = 0$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a) = 0$

According to (5.47), the matrices $W^{(0)}(\mu)$ and $W^{(1)}(\mu)$ respectively defined in (5.105) and (5.96) are the associated boundary matrices of this problem. Choosing $C_2(\mu) = \mathrm{diag}(1, \mu, \mu^3, \mu^5)$, then $C_2(\mu)^{-1}W^{(0)}(\mu)$ gives the matrix obtained in (5.106), since the first two diagonal entries of the matrix $C_2(\mu)$ are identical to the first two diagonal entries of the matrix $C_2(\mu)$ defined in Subsection 5.5.1; while $C_2(\mu)^{-1}W^{(1)}(\mu)$ is the matrix defined in (5.98), because the last two diagonal entries of the matrix $C_2(\mu)$ are identical to the last two diagonal entries of the matrix $C_2(\mu)$ of Subsection 5.4.2. Thus $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39).

### 5.5.3 The boundary conditions $B_3y = y'(a) - i\alpha\lambda y''(a) = 0$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a) = 0$

The associated boundary matrices of this problem are, the matrices $W^{(0)}(\mu)$ and $W^{(1)}(\mu)$ as respectively defined in (5.105) and in (5.99). Choosing $C_2(\mu) = \mathrm{diag}(1, \mu, \mu^4, \mu^5)$, then $C_2(\mu)^{-1}W^{(0)}(\mu) = W_0^{(0)}$, where $W_0^{(0)}$ is the matrix given in (5.106), as the first two diagonal entries of the matrix $C_2(\mu)$ are the same as the first diagonal entries of the matrix $C_2(\mu)$ defined in Subsection 5.5.1; while $C_2(\mu)^{-1}W^{(1)}(\mu)$ produces the matrix given in (5.101), since the last two diagonal entries of the matrix $C_2(\mu)$ are identical to the last two diagonal entries of the matrix $C_2(\mu)$ given in Subsection 5.4.3. Hence $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39).

### 5.5.4 The boundary conditions $B_3y = y'(a) - i\alpha\lambda y''(a) = 0$ and $B_4y = y^{(3)}(a) - i\alpha\lambda y(a) = 0$

The matrix $W^{(0)}(\mu)$ given in (5.105) and the matrix $W^{(1)}(\mu)$ defined in (5.102) are the boundary matrices associated to this problem. Choosing $C_2(\mu) = \operatorname{diag}(1, \mu, \mu^4, \mu^3)$, then $C_2(\mu)^{-1}W^{(0)}(\mu) = W_0^{(0)}$, where $W_0^{(0)}$ is the matrix obtained in (5.106), as the first two diagonal entries of the matrix $C_2(\mu)$ are the same as for the matrix $C_2(\mu)$ defined in Subsection 5.5.1, while $C_2(\mu)^{-1}W^{(1)}(\mu)$ is the matrix given in (5.104), since the last two diagonal entries of the matrix $C_2(\mu)$ are identical two the last two diagonal entries of the matrix $C_2(\mu)$ of Subsection 5.4.4. It follows that $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39).

## 5.6 Birkhoff regularity for $B_1(y) = y^{(3)}(0) = 0$ and $B_2(y) = y''(0) = 0$

The following boundary matrix is associated to all the boundary eigenvalue problems of this section

$$W^{(0)}(\mu) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} C(0,\mu) = \begin{pmatrix} \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ \mu^3 & -i\mu^3 & -\mu^3 & i\mu^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.107}$$

### 5.6.1 The boundary conditions $B_3y = y''(a) + i\alpha\lambda y'(a) = 0$ and $B_4y = y^{(3)}(a) - i\alpha\lambda y(a) = 0$

According to (5.47) the boundary matrix associated to this problem are the matrix $W^{(1)}(\mu)$ defined in (5.87). Choosing $C_2(\mu) = \operatorname{diag}(\mu^2, \mu^3, \mu^3, \mu^3)$, then

$$C_2(\mu)^{-1}W^{(0)}(\mu) = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = W_0^{(0)} \tag{5.108}$$

while $C_2(\mu)^{-1}W^{(1)}(\mu)$ is the matrix obtained in (5.90), since the last two diagonal entries of the matrix $C_2(\mu)$ are identical to the last two diagonal entries of the matrix $C_2(\mu)$ of Subsection 5.4.1. Thus $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39). Hence $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39) and the problem (3.1)–(3.2) is Birkhoff regular for $\alpha > 0$.

### 5.6.2 The boundary conditions $B_3y = y''(a) + i\alpha\lambda y'(a) = 0$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a) = 0$

The boundary matrices associated to this problem are, according to (5.47), the matrix $W^{(0)}(\mu)$ defined in (5.107) and the matrix $W^{(1)}(\mu)$ given by (5.96). Let $C_2(\mu) = \operatorname{diag}(\mu^2, \mu^3, \mu^3, \mu^5)$. The first two diagonal entries of the matrix $C_2(\mu)$ are identical to the first two diagonal entries of the matrix $C_2(\mu)$ of Subsection 5.6.1, then $C_2(\mu)^{-1}W^{(0)}(\mu)$ is the matrix given in (5.108); on the other hand, the last two diagonal entries of the matrix $C_2(\mu)$ are the same as for the matrix $C_2(\mu)$ Subsection 5.4.2, thus $C_2(\mu)^{-1}W^{(1)}(\mu)$ is the matrix yielded in (5.98) . It follows that $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39).

### 5.6.3 The boundary conditions $B_3y = y'(a) - i\alpha\lambda y''(a) = 0$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a) = 0$

The matrix $W^{(0)}(\mu)$ given in (5.107) and the matrix $W^{(1)}(\mu)$ defined in (5.99) are the boundary matrices associated to this problem. Choosing $C_2(\mu) = \mathrm{diag}(\mu^2, \mu^3, \mu^4, \mu^5)$, then $C_2(\mu)^{-1}W^{(0)}(\mu)$ is the matrix yielded in (5.108), because the first two diagonal entries of the matrix $C_2(\mu)$ are identical to the first two diagonal entries of the matrix $C_2(\mu)$ of Subsection 5.6.1; while $C_2(\mu)^{-1}W^{(1)}(\mu)$ is the matrix obtained in (5.101), as the last two diagonal entries of the matrix $C_2(\mu)$ are the same as the last two diagonal entries of the matrix $C_2(\mu)$ of Subsection 5.4.3. Thus $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39).

### 5.6.4 The boundary conditions $B_3y = y'(a) - i\alpha\lambda y''(a) = 0$ and $B_4y = y^{(3)}(a) - i\alpha\lambda y(a) = 0$

The boundary matrices associated to this problem are the matrix $W^{(0)}(\mu)$ given in (5.107) and the matrix $W^{(1)}(\mu)$ defined in (5.102). Let $C_2(\mu) = \mathrm{diag}(\mu^2, \mu^3, \mu^4, \mu^3)$. The first two diagonal entries of the matrix $C_2(\mu)$ are identical to the first two diagonal entries of the matrix $C_2(\mu)$ of Subsection 5.6.1, then $C_2(\mu)^{-1}W^{(0)}(\mu)$ is the matrix obtained in (5.108). Since the last two diagonal entries of the matrix $C_2(\mu)$ are the same as for the matrix $C_2(\mu)$ of Subsection 5.4.4, then $C_2(\mu)^{-1}W^{(1)}(\mu)$ is the matrix given in (5.104). It follows that $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39).

## 5.7 Birkhoff regularity for $B_1(y) = y^{(3)}(0) = 0$ and $B_2(y) = y'(0) = 0$

The following matrix is associated to all the boundary eigenvalue problems of this section

$$W^{(0)}(\mu) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} C(0,\mu) = \begin{pmatrix} \mu & i\mu & -\mu & -i\mu \\ \mu^3 & -i\mu^3 & -\mu^3 & i\mu^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.109}$$

### 5.7.1 The boundary conditions $B_3y = y''(a) + i\alpha\lambda y'(a) = 0$ and $B_4y = y^{(3)}(a) - i\alpha\lambda y(a) = 0$

According to (5.47) the second boundary matrix associated to this problem is the matrix $W^{(1)}(\mu)$ defined in (5.87). Choosing $C_2(\mu) = \operatorname{diag}(\mu, \mu^3, \mu^3, \mu^3)$, then

$$C_2(\mu)^{-1}W^{(0)}(\mu) = \begin{pmatrix} 1 & i & -1 & -i \\ 1 & -i & -1 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = W_0^{(0)}, \tag{5.110}$$

and since the last two diagonal entries of the matrix $C_2(\mu)$ are identical to the last two diagonal entries of the matrix $C_2(\mu)$ of Subsection 5.4.1, $C_2(\mu)^{-1}W^{(1)}(\mu)$ is as obtained in (5.90). Thus $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39) and it is easy to check that (5.35) and (5.36) hold.

### 5.7.2 The boundary conditions $B_3y = y''(a) + i\alpha\lambda y'(a) = 0$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a) = 0$

The associated boundary matrices of this problem are, according to (5.47), the matrix $W^{(0)}(\mu)$ defined in (5.109) and the matrix $W^{(1)}(\mu)$ given by (5.96). Let $C_2(\mu) = \operatorname{diag}(\mu, \mu^3, \mu^3, \mu^5)$. The first two diagonal entries of the matrix $C_2(\mu)$ are the same as for the matrix $C_2(\mu)$ of Subsection 5.7.1, then $C_2(\mu)^{-1}W^{(0)}(\mu)$ is the matrix defined (5.110); on the other hand the last two diagonal entries of $C_2(\mu)$ are identical to the last two diagonal entries of the matrix $C_2(\mu)$ of Subsection 5.4.2, thus $C_2(\mu)^{-1}W^{(1)}(\mu)$ is the matrix given in (5.98). Hence $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39).

### 5.7.3 The boundary conditions $B_3y = y'(a) - i\alpha\lambda y''(a) = 0$ and $B_4y = y(a) + i\alpha\lambda y^{(3)}(a) = 0$

The matrix $W^{(0)}(\mu)$ given in (5.109) and the matrix $W^{(1)}(\mu)$ defined in (5.99) are the boundary matrices associated to this problem. Choosing $C_2(\mu) = \mathrm{diag}(\mu, \mu^3, \mu^4, \mu^5)$, then $C_2(\mu)^{-1}W^{(0)}(\mu)$ is the matrix given in (5.110), since the first two diagonal entries of the matrix $C_2(\mu)$ are identical to the the first two diagonal entries of the matrix $C_2(\mu)$ of Subsection 5.7.1, while $C_2(\mu)^{-1}W^{(1)}(\mu)$ is the matrix obtained in (5.101), because the last two diagonal entries of the matrix $C_2(\mu)$ are the same as for the matrix $C_2(\mu)$ of Subsection 5.4.3. Thus $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39).

### 5.7.4 The boundary conditions $B_3y = y'(a) - i\alpha\lambda y''(a) = 0$ and $B_4y = y^{(3)}(a) - i\alpha\lambda y(a) = 0$

The boundary matrices associated to this problem are the matrix $W^{(0)}(\mu)$ given in (5.109) and the matrix $W^{(1)}(\mu)$ defined in (5.102). Let $C_2(\mu) = \mathrm{diag}(\mu^2, \mu^3, \mu^4, \mu^3)$. The first two diagonal entries of $C_2(\mu)$ are the same as the first two diagonal entries of the matrix $C_2(\mu)$ of Subsection 5.7.1, thus $C_2(\mu)^{-1}W^{(0)}(\mu)$ is the matrix obtained in (5.110); on the other hand the last two diagonal entries of $C_2(\mu)$ are identical to the last two diagonal entries of the matrix $C_2(\mu)$ of Subsection 5.4.4, then $C_2(\mu)^{-1}W^{(1)}(\mu)$ is the matrix produced in (5.104). It follows that $C_2(\mu)^{-1}W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$ holds for $j = 0, 1$, see (5.39).

## 5.8 Birkhoff regularity of eigenvalue problem (3.1)–(3.2)

**Remark 5.30.** Consider

$$W_0^{(0)}\Delta_j + W_0^{(1)}(I - \Delta_j), \tag{5.111}$$

where $\Delta_j$, $j = 1, 2, 3, 4$ are the matrices defined in (5.84). It is easy to check that $I - \Delta_1 = \Delta_3$ while $I - \Delta_2 = \Delta_4$, it follows that after a permutation of columns, the matrices (5.91) are block diagonal matrices consisting of $2 \times 2$ blocks taken from two consecutive columns (in the sense of cyclic arrangement) of the first two rows of $W_0^{(0)}$ and the last two rows of $W_0^{(1)}$, respectively for the eigenvalue problems (3.1)–(3.2) defined in each of subsections of Section 5.4 to Section 5.7. These matrices are obviously invertible for $\alpha > 0$ and for all four choices of $\Delta$. Thus Proposition 5.13 and Definition 5.21 yield the following proposition,

**Proposition 5.31.** *The eigenvalue problems* (3.1)–(3.2) *defined in each of subsections of Section 5.4 to Section 5.7 are Birkhoff regular for* $\alpha > 0$.

## 5.9 The canonical fundamental system of $y^{(4)} - (gy')' = \lambda^2 y$

We have $n_0 = 0$, see (5.49)–(5.51), thus $l = 4$, see Theorem 5.23. As $n_0 = 0$, the matrix $D$ is reduced to the null matrix, hence the matrix $M_2(\cdot, \mu)$ defined in (5.68) is reduced to $I_4$. It is easy to check that $\Omega_4 = \mathrm{diag}(1, i, -1, -i)$, see Theorem 5.23, thus $\Omega_4^0 = I_4$. Since $n_0 = 0$, then $n_0 - 1 < 0$ and it follows that $h_{n_0-1} = 0$. Let

$$\widetilde{Q} := \widetilde{Q}_{22}.$$

Since $n = 4$, then $k_3 = k_0 = 0$, $k_2 = -g$ and $k_1 = -g'$, see (5.49)–(5.51). As $k_3 = 0$ and $h_{n_0-1} = 0$, then $Q^{[0]'} = 0$ and $Q^{[0]}(a) = I_4$, see (5.71), thus it follows that

$$Q^{[0]} = I_4. \tag{5.112}$$

According to Theorem 5.23 ii), we have

$$\begin{cases} \omega_1 = 1 \\ \omega_2 = \exp(\frac{2\pi i}{4}) = i \\ \omega_3 = \exp(\pi i) = -1 \\ \omega_4 = \exp(\frac{3\pi i}{2}) = -i \end{cases}. \tag{5.113}$$

Set $\Omega := \Omega_4$. Then $\Omega = \mathrm{diag}(\omega_1, \omega_2, \omega_3, \omega_4) = \mathrm{diag}(1, i, -1, -i)$. Set $Q := Q_{22}$ and

$$Q^{[1]} = \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{14}^1 \\ a_{21}^1 & a_{22}^1 & a_{23}^1 & a_{24}^1 \\ a_{31}^1 & a_{32}^1 & a_{33}^1 & a_{34}^1 \\ a_{41}^1 & a_{42}^1 & a_{43}^1 & a_{44}^1 \end{pmatrix}.$$

Then

$$\Omega Q^{[1]} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{14}^1 \\ a_{21}^1 & a_{22}^1 & a_{23}^1 & a_{24}^1 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{14}^1 \\ ia_{21}^1 & ia_{22}^1 & ia_{23}^1 & ia_{24}^1 \\ -a_{31}^1 & -a_{32}^1 & -a_{33}^1 & -a_{34}^1 \\ -ia_{41}^1 & -ia_{42}^1 & -ia_{43}^1 & -ia_{44}^1 \end{pmatrix}$$

and

$$Q^{[1]}\Omega = \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{14}^1 \\ a_{21}^1 & a_{22}^1 & a_{23}^1 & a_{24}^1 \\ a_{31}^1 & a_{32}^1 & a_{33}^1 & a_{34}^1 \\ a_{41}^1 & a_{42}^1 & a_{43}^1 & a_{44}^1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} = \begin{pmatrix} a_{11}^1 & ia_{12}^1 & -a_{13}^1 & -ia_{14}^1 \\ a_{21}^1 & ia_{22}^1 & -a_{23}^1 & -ia_{24}^1 \\ a_{31}^1 & ia_{32}^1 & -a_{33}^1 & -ia_{34}^1 \\ a_{41}^1 & ia_{42}^1 & -a_{43}^1 & -ia_{44}^1 \end{pmatrix}.$$

Thus

$$\begin{aligned} \Omega Q^{[1]} - Q^{[1]}\Omega &= \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{14}^1 \\ ia_{21}^1 & ia_{22}^1 & ia_{23}^1 & ia_{24}^1 \\ -a_{31}^1 & -a_{32}^1 & -a_{33}^1 & -a_{34}^1 \\ -ia_{41}^1 & -ia_{42}^1 & -ia_{43}^1 & -ia_{44}^1 \end{pmatrix} - \begin{pmatrix} a_{11}^1 & ia_{12}^1 & -a_{13}^1 & -ia_{14}^1 \\ a_{21}^1 & ia_{22}^1 & -a_{23}^1 & -ia_{24}^1 \\ a_{31}^1 & ia_{32}^1 & -a_{33}^1 & -ia_{34}^1 \\ a_{41}^1 & ia_{42}^1 & -a_{43}^1 & -ia_{44}^1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & (1-i)a_{12}^1 & 2a_{13}^1 & (1+i)a_{14}^1 \\ (-1+i)a_{21}^1 & 0 & (1+i)a_{23}^1 & 2ia_{24}^1 \\ -2a_{31}^1 & -(1+i)a_{32}^1 & 0 & (-1+i)a_{34}^1 \\ -(1+i)a_{41}^1 & -2ia_{42}^1 & (1-i)a_{43}^1 & 0 \end{pmatrix}. \end{aligned} \tag{5.114}$$

But it follows from (5.76) that $\Omega Q^{[1]} - Q^{[1]}\Omega = Q^{[0]\prime}$, thus $\Omega Q^{[1]} - Q^{[1]}\Omega = 0$, see (5.112). It results from (5.114) that $a_{12}^1 = a_{13}^1 = a_{14}^1 = 0$, $a_{21}^1 = a_{23}^1 = a_{24}^1 = 0$, $a_{31}^1 = a_{32}^1 = a_{34}^1 = 0$ and $a_{41}^1 = a_{42}^1 = a_{43}^1 = 0$, so

$$Q^{[1]} = \begin{pmatrix} a_{11}^1 & 0 & 0 & 0 \\ 0 & a_{22}^1 & 0 & 0 \\ 0 & 0 & a_{33}^1 & 0 \\ 0 & 0 & 0 & a_{44}^1 \end{pmatrix}. \tag{5.115}$$

Set

$$Q^{[2]} = \begin{pmatrix} a^2_{11} & a^2_{12} & a^2_{13} & a^2_{14} \\ a^2_{21} & a^2_{22} & a^2_{23} & a^2_{24} \\ a^2_{31} & a^2_{32} & a^2_{33} & a^2_{34} \\ a^2_{41} & a^2_{42} & a^2_{43} & a^2_{44} \end{pmatrix}.$$

It follows from (5.76) that $\Omega Q^{[2]} - Q^{[2]}\Omega = Q^{[1]'} - \frac{1}{4}g\Omega\varepsilon\varepsilon^\top\Omega^{-2}Q^{[0]}$ and as in (5.114) that

$$\Omega Q^{[2]} - Q^{[2]}\Omega = \begin{pmatrix} 0 & (1-i)a^2_{12} & 2a^2_{13} & (1+i)a^2_{14} \\ (-1+i)a^2_{21} & 0 & (1+i)a^2_{23} & 2ia^2_{24} \\ -2a^2_{31} & -(1+i)a^2_{32} & 0 & (-1+i)a^2_{34} \\ -(1+i)a^2_{41} & -2ia^2_{42} & (1-i)a^2_{43} & 0 \end{pmatrix}. \tag{5.116}$$

It is easy to check that

$$\varepsilon\varepsilon^\top = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},\ \Omega^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \text{ and } \Omega^{-2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5.117}$$

Thus we have

$$\varepsilon\varepsilon^\top\Omega^{-2} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \tag{5.118}$$

and

$$\Omega\varepsilon\varepsilon^\top\Omega^{-2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ i & -i & i & -i \\ -1 & 1 & -1 & 1 \\ -i & i & -i & i \end{pmatrix}. \tag{5.119}$$

Hence

$$Q^{[1]'} - \frac{1}{4}g\Omega\varepsilon\varepsilon^\top\Omega^{-2}I_4 = \begin{pmatrix} a^{1'}_{11} - \frac{1}{4}g & \frac{1}{4}g & -\frac{1}{4}g & \frac{1}{4}g \\ -\frac{i}{4}g & a^{1'}_{22} + \frac{i}{4}g & -\frac{i}{4}g & \frac{i}{4}g \\ \frac{1}{4}g & -\frac{1}{4}g & a^{1'}_{33} + \frac{1}{4}g & -\frac{1}{4}g \\ \frac{i}{4}g & -\frac{i}{4}g & \frac{i}{4}g & a^{1'}_{44} - \frac{i}{4}g \end{pmatrix}. \tag{5.120}$$

Let $G(x) = \int_0^x g(t)dt$. Then it follows from (5.116) and (5.120) that

$$\begin{cases} a_{12}^2 = \frac{1+i}{8} g \\ a_{13}^2 = -\frac{1}{8} g \\ a_{14}^2 = \frac{1-i}{8} g \end{cases} \quad \begin{cases} a_{21}^2 = \frac{-1+i}{8} g \\ a_{23}^2 = -\frac{1+i}{8} g \\ a_{24}^2 = \frac{1}{8} g \end{cases} \quad \begin{cases} a_{31}^2 = -\frac{1}{8} g \\ a_{32}^2 = \frac{1-i}{8} g \\ a_{34}^2 = \frac{1+i}{8} g \end{cases} \quad \begin{cases} a_{41}^2 = -\frac{1+i}{8} g \\ a_{42}^2 = \frac{1}{8} g \\ a_{43}^2 = \frac{-1+i}{8} g \end{cases} \tag{5.121}$$

and

$$a_{11}^1 = \frac{1}{4} G(x), \quad a_{22}^1 = -\frac{i}{4} G(x), \quad a_{33}^1 = -\frac{1}{4} G(x), \quad a_{44}^1 = \frac{i}{4} G(x). \tag{5.122}$$

Hence

$$Q^{[1]} = \begin{pmatrix} \frac{1}{4} G & 0 & 0 & 0 \\ 0 & -\frac{i}{4} G & 0 & 0 \\ 0 & 0 & -\frac{1}{4} G & 0 \\ 0 & 0 & 0 & \frac{i}{4} G \end{pmatrix}, \tag{5.123}$$

see (5.115) and

$$Q^{[2]} = \begin{pmatrix} a_{11}^2 & \frac{1+i}{8} g & -\frac{1}{8} g & \frac{1-i}{8} g \\ \frac{-1+i}{8} g & a_{22}^2 & -\frac{1+i}{8} g & \frac{1}{8} g \\ -\frac{1}{8} g & \frac{1-i}{8} g & a_{33}^2 & \frac{1+i}{8} g \\ -\frac{1+i}{8} g & \frac{1}{8} g & \frac{-1+i}{8} g & a_{44}^2 \end{pmatrix}. \tag{5.124}$$

Set

$$Q^{[3]} = \begin{pmatrix} a_{11}^3 & a_{12}^3 & a_{13}^3 & a_{14}^3 \\ a_{21}^3 & a_{22}^3 & a_{23}^3 & a_{24}^3 \\ a_{31}^3 & a_{32}^3 & a_{33}^3 & a_{34}^3 \\ a_{41}^3 & a_{42}^3 & a_{43}^3 & a_{44}^3 \end{pmatrix}.$$

It follows from (5.76) that $\Omega Q^{[3]} - Q^{[3]}\Omega = Q^{[2]\prime} - \frac{1}{4}\left(g\Omega\varepsilon\varepsilon^\top\Omega^{-2}Q^{[1]} + g'\Omega\varepsilon\varepsilon^\top\Omega^{-3}Q^{[0]}\right)$ and as in (5.114) that

$$\Omega Q^{[3]} - Q^{[3]}\Omega = \begin{pmatrix} 0 & (1-i)a_{12}^3 & 2a_{13}^3 & (1+i)a_{14}^3 \\ (-1+i)a_{21}^3 & 0 & (1+i)a_{23}^3 & 2ia_{24}^3 \\ -2a_{31}^3 & -(1+i)a_{32}^3 & 0 & (-1+i)a_{34}^3 \\ -(1+i)a_{41}^3 & -2ia_{42}^3 & (1-i)a_{43}^3 & 0 \end{pmatrix} \tag{5.125}$$

It can be inferred from (5.119) and (5.123) that

$$\Omega\varepsilon\varepsilon^\top\Omega^{-2}Q^{[1]} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ i & -i & i & -i \\ -1 & 1 & -1 & 1 \\ -i & i & -i & i \end{pmatrix} \begin{pmatrix} \frac{1}{4}G & 0 & 0 & 0 \\ 0 & -\frac{i}{4}G & 0 & 0 \\ 0 & 0 & -\frac{1}{4}G & 0 \\ 0 & 0 & 0 & \frac{i}{4}G \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}G & \frac{i}{4}G & -\frac{1}{4}G & -\frac{i}{4}G \\ \frac{i}{4}G & -\frac{1}{4}G & -\frac{i}{4}G & \frac{1}{4}G \\ -\frac{1}{4}G & -\frac{i}{4}G & \frac{1}{4}G & \frac{i}{4}G \\ -\frac{i}{4}G & \frac{1}{4}G & \frac{i}{4}G & -\frac{1}{4}G \end{pmatrix} \tag{5.126}$$

and from (5.117) and (5.119)

$$\Omega\varepsilon\varepsilon^\top\Omega^{-3} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ i & -i & i & -i \\ -1 & 1 & -1 & 1 \\ -i & i & -i & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} = \begin{pmatrix} 1 & i & -1 & -i \\ i & -1 & -i & 1 \\ -1 & -i & 1 & i \\ -i & 1 & i & -1 \end{pmatrix}. \tag{5.127}$$

Let $K := g\Omega\varepsilon\varepsilon^\top\Omega^{-2}Q^{[1]} + g'\Omega\varepsilon\varepsilon^\top\Omega^{-3}I_4$ and $\widetilde{K} := Q^{[2]'} - \frac{1}{4}\left(g\Omega\varepsilon\varepsilon^\top\Omega^{-2}Q^{[1]} + g'\Omega\varepsilon\varepsilon^\top\Omega^{-3}I_4\right)$. Then it results from (5.126) and (5.127) that

$$K = \begin{pmatrix} \frac{1}{4}gG & \frac{i}{4}gG & -\frac{1}{4}gG & -\frac{i}{4}gG \\ \frac{i}{4}gG & -\frac{1}{4}gG & -\frac{i}{4}gG & \frac{1}{4}gG \\ -\frac{1}{4}gG & -\frac{i}{4}gG & \frac{1}{4}gG & \frac{i}{4}gG \\ -\frac{i}{4}gG & \frac{1}{4}gG & \frac{i}{4}gG & -\frac{1}{4}gG \end{pmatrix} + \begin{pmatrix} g' & ig' & -g' & -ig' \\ ig' & -g' & -ig' & g' \\ -g' & -ig' & g' & ig' \\ -ig' & g' & ig' & -g' \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}gG + g' & \frac{i}{4}gG + ig' & -\frac{1}{4}gG - g' & -\frac{i}{4}gG - ig' \\ \frac{i}{4}gG + ig' & -\frac{1}{4}gG - g' & -\frac{i}{4}gG - ig' & \frac{1}{4}gG + g' \\ -\frac{1}{4}gG - g' & -\frac{i}{4}gG - ig' & \frac{1}{4}gG + g' & \frac{i}{4}gG + ig' \\ -\frac{i}{4}gG - ig' & \frac{1}{4}gG + g' & \frac{i}{4}gG + ig' & -\frac{1}{4}gG - g' \end{pmatrix} \tag{5.128}$$

and from (5.124) and (5.128)

$$\widetilde{K} = \begin{pmatrix} a_{11}^{2'} - \frac{1}{16}gG - \frac{1}{4}g' & \frac{1-i}{8}g' - \frac{i}{16}gG & \frac{1}{8}g' + \frac{1}{16}gG & \frac{1+i}{8}g' + \frac{i}{16}gG \\ -\frac{1+i}{8}g' - \frac{i}{16}gG & a_{22}^{2'} + \frac{1}{16}gG + \frac{1}{4}g' & -\frac{1-i}{8}g' + \frac{i}{16}gG & -\frac{1}{8}g' - \frac{1}{16}gG \\ \frac{1}{8}g' + \frac{1}{16}gG & \frac{1+i}{8}g' + \frac{i}{16}gG & a_{33}^{2'} - \frac{1}{16}gG - \frac{1}{4}g' & \frac{1-i}{8}g' - \frac{i}{16}gG \\ -\frac{1-i}{8}g' + \frac{i}{16}gG & -\frac{1}{8}g' - \frac{1}{16}gG & -\frac{1+i}{8}g' - \frac{i}{16}gG & a_{44}^{2'} + \frac{1}{16}gG + \frac{1}{4}g' \end{pmatrix}. \tag{5.129}$$

Thus it follows from (5.125) and (5.129) that

$$a_{11}^2 = \frac{1}{32}G^2 + \frac{1}{4}g \quad a_{22}^2 = -\frac{1}{32}G^2 - \frac{1}{4}g \quad a_{33}^2 = \frac{1}{32}G^2 + \frac{1}{4}g \quad a_{44}^2 = -\frac{1}{32}G^2 - \frac{1}{4}g \tag{5.130}$$

and

$$\begin{cases} a_{12}^3 = \frac{1}{8}g' + \frac{1-i}{32}gG \\ a_{13}^3 = \frac{1}{16}g' + \frac{1}{32}gG \\ a_{14}^3 = \frac{1}{8}g' + \frac{1+i}{32}gG \end{cases} \qquad \begin{cases} a_{21}^3 = \frac{i}{8}g' - \frac{1-i}{32}gG \\ a_{23}^3 = \frac{i}{8}g' + \frac{1+i}{32}gG \\ a_{24}^3 = \frac{i}{16}g' + \frac{i}{32}gG \end{cases} \tag{5.131}$$

$$\begin{cases} a_{31}^3 = -\frac{1}{16}g' - \frac{1}{32}gG \\ a_{32}^3 = -\frac{1}{8}g' - \frac{1+i}{32}gG \\ a_{34}^3 = -\frac{1}{8}g' - \frac{1-i}{32}gG \end{cases} \qquad \begin{cases} a_{41}^3 = -\frac{i}{8}g' - \frac{1+i}{32}gG \\ a_{42}^3 = -\frac{i}{16}g' - \frac{i}{32}gG \\ a_{43}^3 = -\frac{i}{8}g' + \frac{1-i}{32}gG \end{cases}. \tag{5.132}$$

Hence

$$Q^{[3]} = \begin{pmatrix} a_{11}^3 & \frac{1}{8}g' + \frac{1-i}{32}gG & \frac{1}{16}g' + \frac{1}{32}gG & \frac{1}{8}g' + \frac{1+i}{32}gG \\ \frac{i}{8}g' - \frac{1-i}{32}gG & a_{22}^3 & \frac{i}{8}g' + \frac{1+i}{32}gG & \frac{i}{16}g' + \frac{i}{32}gG \\ -\frac{1}{16}g' - \frac{1}{32}gG & -\frac{1}{8}g' - \frac{1+i}{32}gG & a_{33}^3 & -\frac{1}{8}g' - \frac{1-i}{32}gG \\ -\frac{i}{8}g' - \frac{1+i}{32}gG & -\frac{i}{16}g' - \frac{i}{32}gG & -\frac{i}{8}g' + \frac{1-i}{32}gG & a_{44}^3 \end{pmatrix} \tag{5.133}$$

and it follows from (5.124) and (5.130) that

$$Q^{[2]} = \begin{pmatrix} \frac{1}{32}G^2 + \frac{1}{4}g & \frac{1+i}{8}g & -\frac{1}{8}g & \frac{1-i}{8}g \\ \frac{-1+i}{8}g & -\frac{1}{32}G^2 - \frac{1}{4}g & -\frac{1+i}{8}g & \frac{1}{8}g \\ -\frac{1}{8}g & \frac{1-i}{8}g & \frac{1}{32}G^2 + \frac{1}{4}g & \frac{1+i}{8}g \\ -\frac{1+i}{8}g & \frac{1}{8}g & \frac{-1+i}{8}g & -\frac{1}{32}G^2 - \frac{1}{4}g \end{pmatrix}. \tag{5.134}$$

Set

$$Q^{[4]} = \begin{pmatrix} a_{11}^4 & a_{12}^4 & a_{13}^4 & a_{14}^4 \\ a_{21}^4 & a_{22}^4 & a_{23}^4 & a_{24}^4 \\ a_{31}^4 & a_{32}^4 & a_{33}^4 & a_{34}^4 \\ a_{41}^4 & a_{42}^4 & a_{43}^4 & a_{44}^4 \end{pmatrix}.$$

It follows from (5.76) that

$$\Omega Q^{[4]} - Q^{[4]}\Omega = Q^{[3]\prime} - \frac{1}{4}\left(g\Omega\varepsilon\varepsilon^{\top}\Omega^{-2}Q^{[2]} + g'\Omega\varepsilon\varepsilon^{\top}\Omega^{-3}Q^{[1]}\right) \tag{5.135}$$

and as in (5.114) that

$$\Omega Q^{[4]} - Q^{[4]}\Omega = \begin{pmatrix} 0 & (1-i)a_{12}^4 & 2a_{13}^4 & (1+i)a_{14}^4 \\ (-1+i)a_{21}^4 & 0 & (1+i)a_{23}^4 & 2ia_{24}^4 \\ -2a_{31}^4 & -(1+i)a_{32}^4 & 0 & (-1+i)a_{34}^4 \\ -(1+i)a_{41}^4 & -2ia_{42}^4 & (1-i)a_{43}^4 & 0 \end{pmatrix}. \tag{5.136}$$

Using (5.119) and (5.134), we have

$$\begin{aligned} \Omega\varepsilon\varepsilon^{\top}\Omega^{-2}Q^{[2]} &= \begin{pmatrix} 1 & -1 & 1 & -1 \\ i & -i & i & -i \\ -1 & 1 & -1 & 1 \\ -i & i & -i & i \end{pmatrix} \begin{pmatrix} \frac{1}{32}G^2 + \frac{1}{4}g & \frac{1+i}{8}g & -\frac{1}{8}g & \frac{1-i}{8}g \\ \frac{-1+i}{8}g & -\frac{1}{32}G^2 - \frac{1}{4}g & -\frac{1+i}{8}g & \frac{1}{8}g \\ -\frac{1}{8}g & \frac{1-i}{8}g & \frac{1}{32}G^2 + \frac{1}{4}g & \frac{1+i}{8}g \\ -\frac{1+i}{8}g & \frac{1}{8}g & \frac{-1+i}{8}g & -\frac{1}{32}G^2 - \frac{1}{4}g \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{32}G^2 + \frac{3}{8}g & \frac{1}{32}G^2 + \frac{3}{8}g & \frac{1}{32}G^2 + \frac{3}{8}g & \frac{1}{32}G^2 + \frac{3}{8}g \\ \frac{i}{32}G^2 + \frac{3i}{8}g & \frac{i}{32}G^2 + \frac{3i}{8}g & \frac{i}{32}G^2 + \frac{3i}{8}g & \frac{i}{32}G^2 + \frac{3i}{8}g \\ -\frac{1}{32}G^2 - \frac{3}{8}g & -\frac{1}{32}G^2 - \frac{3}{8}g & -\frac{1}{32}G^2 - \frac{3}{8}g & -\frac{1}{32}G^2 - \frac{3}{8}g \\ -\frac{i}{32}G^2 - \frac{3i}{8}g & -\frac{i}{32}G^2 - \frac{3i}{8}g & -\frac{i}{32}G^2 - \frac{3i}{8}g & -\frac{i}{32}G^2 - \frac{3i}{8}g \end{pmatrix}, \end{aligned} \tag{5.137}$$

while (5.123) and (5.127) yield

$$
\begin{aligned}
\Omega\varepsilon\varepsilon^{\top}\Omega^{-3}Q^{[1]} &= \begin{pmatrix} 1 & i & -1 & -i \\ i & -1 & -i & 1 \\ -1 & -i & 1 & i \\ -i & 1 & i & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{4}G & 0 & 0 & 0 \\ 0 & -\frac{i}{4}G & 0 & 0 \\ 0 & 0 & -\frac{1}{4}G & 0 \\ 0 & 0 & 0 & \frac{i}{4}G \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{4}G & \frac{1}{4}G & \frac{1}{4}G & \frac{1}{4}G \\ \frac{i}{4}G & \frac{i}{4}G & \frac{i}{4}G & \frac{i}{4}G \\ -\frac{1}{4}G & -\frac{1}{4}G & -\frac{1}{4}G & -\frac{1}{4}G \\ -\frac{i}{4}G & -\frac{i}{4}G & -\frac{i}{4}G & -\frac{i}{4}G \end{pmatrix}. 
\end{aligned} \tag{5.138}
$$

We require, for the remainder of the chapter, that $g \in C^2[0,a]$, where $a > 0$. Let $K_1 := \frac{1}{4}\left(g\Omega\varepsilon\varepsilon^{\top}\Omega^{-2}Q^{[2]} + g'\Omega\varepsilon\varepsilon^{\top}\Omega^{-3}Q^{[1]}\right)$ and $\widetilde{K}_1 := Q^{[3]'} - \frac{1}{4}\left(g\Omega\varepsilon\varepsilon^{\top}\Omega^{-2}Q^{[2]} + g'\Omega\varepsilon\varepsilon^{\top}\Omega^{-3}Q^{[1]}\right)$. Then it follows from (5.137) and (5.138) that

$$
\begin{aligned}
K_1 &= \begin{pmatrix} \frac{1}{128}gG^2 + \frac{3}{32}g^2 & \frac{1}{128}gG^2 + \frac{3}{32}g^2 & \frac{1}{128}gG^2 + \frac{3}{32}g^2 & \frac{1}{128}gG^2 + \frac{3}{32}g^2 \\ \frac{i}{128}gG^2 + \frac{3i}{32}g^2 & \frac{i}{128}gG^2 + \frac{3i}{32}g^2 & \frac{i}{128}gG^2 + \frac{3i}{32}g^2 & \frac{i}{128}gG^2 + \frac{3i}{32}g^2 \\ -\frac{1}{128}gG^2 - \frac{3}{32}g^2 & -\frac{1}{128}gG^2 - \frac{3}{32}g^2 & -\frac{1}{128}gG^2 - \frac{3}{32}g^2 & -\frac{1}{128}gG^2 - \frac{3}{32}g^2 \\ -\frac{i}{128}gG^2 - \frac{3i}{32}g^2 & -\frac{i}{128}gG^2 - \frac{3i}{32}g^2 & -\frac{i}{128}gG^2 - \frac{3i}{32}g^2 & -\frac{i}{128}gG^2 - \frac{3i}{32}g^2 \end{pmatrix} \\
&+ \begin{pmatrix} \frac{1}{16}g'G & \frac{1}{16}g'G & \frac{1}{16}g'G & \frac{1}{16}g'G \\ \frac{i}{16}g'G & \frac{i}{16}g'G & \frac{i}{16}G & \frac{i}{16}g'G \\ -\frac{1}{16}g'G & -\frac{1}{16}g'G & -\frac{1}{16}g'G & -\frac{1}{16}g'G \\ -\frac{i}{16}g'G & -\frac{i}{16}g'G & -\frac{i}{16}g'G & -\frac{i}{16}g'G \end{pmatrix} \\
&= \left( \begin{matrix} \frac{1}{128}gG^2 + \frac{3}{32}g^2 + \frac{1}{16}g'G & \frac{1}{128}gG^2 + \frac{3}{32}g^2 + \frac{1}{16}g'G \\ \frac{i}{128}gG^2 + \frac{3i}{32}g^2 + \frac{i}{16}g'G & \frac{i}{128}gG^2 + \frac{3i}{32}g^2 + \frac{i}{16}g'G \\ -\frac{1}{128}gG^2 - \frac{3}{32}g^2 - \frac{1}{16}g'G & -\frac{1}{128}gG^2 - \frac{3}{32}g^2 - \frac{1}{16}g'G \\ -\frac{i}{128}gG^2 - \frac{3i}{32}g^2 - \frac{i}{16}g'G & -\frac{i}{128}gG^2 - \frac{3i}{32}g^2 - \frac{i}{16}g'G \end{matrix} \right. \\
&\qquad \left. \begin{matrix} \frac{1}{128}gG^2 + \frac{3}{32}g^2 + \frac{1}{16}g'G & \frac{1}{128}gG^2 + \frac{3}{32}g^2 + \frac{1}{16}g'G \\ \frac{i}{128}gG^2 + \frac{3i}{32}g^2 + \frac{i}{16}g'G & \frac{i}{128}gG^2 + \frac{3i}{32}g^2 + \frac{i}{16}g'G \\ -\frac{1}{128}gG^2 - \frac{3}{32}g^2 - \frac{1}{16}g'G & -\frac{1}{128}gG^2 - \frac{3}{32}g^2 - \frac{1}{16}g'G \\ -\frac{i}{128}gG^2 - \frac{3i}{32}g^2 - \frac{i}{16}g'G & -\frac{i}{128}gG^2 - \frac{3i}{32}g^2 - \frac{i}{16}g'G \end{matrix} \right),
\end{aligned} \tag{5.139}
$$

while (5.133) and (5.139) give

$$\widetilde{K}_1 = \left(\begin{matrix} a_{11}^{3\prime} - \frac{1}{128}gG^2 - \frac{3}{32}g^2 - \frac{1}{16}g'G & \frac{1}{8}g'' - \frac{1+i}{32}g'G - \frac{2+i}{32}g^2 - \frac{1}{128}gG^2 \\ \frac{i}{8}g'' - \frac{1+i}{32}g'G - \frac{1+2i}{32}g^2 - \frac{i}{128}gG^2 & a_{22}^{3\prime} - \frac{i}{128}gG^2 - \frac{3i}{32}g^2 - \frac{i}{16}g'G \\ -\frac{1}{16}g'' + \frac{1}{16}g^2 + \frac{1}{32}g'G + \frac{1}{128}gG^2 & -\frac{1}{8}g'' + \frac{1-i}{32}g'G + \frac{2-i}{32}g^2 + \frac{1}{128}gG^2 \\ -\frac{i}{8}g'' - \frac{1-i}{32}g'G - \frac{1-2i}{32}g^2 + \frac{i}{128}gG^2 & -\frac{i}{16}g'' + \frac{i}{32}g'G + \frac{i}{16}g^2 + \frac{i}{128}gG^2 \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{1}{16}g'' - \frac{1}{32}g'G - \frac{1}{16}g^2 - \frac{1}{128}gG^2 & \frac{1}{8}g'' - \frac{1-i}{32}g'G - \frac{2-i}{32}g^2 - \frac{1}{128}gG^2 \\ \frac{i}{8}g'' + \frac{1-i}{32}g'G + \frac{1-2i}{32}g^2 - \frac{i}{128}gG^2 & \frac{i}{16}g'' - \frac{i}{32}g'G - \frac{i}{16}g^2 - \frac{i}{128}gG^2 \\ c_{33}' + \frac{3}{32}g^2 + \frac{1}{16}g'G + \frac{1}{128}gG^2 & -\frac{1}{8}g'' + \frac{1+i}{32}g'G + \frac{2+i}{32}g^2 + \frac{1}{128}gG^2 \\ -\frac{i}{8}g'' + \frac{1+i}{32}g'G + \frac{1+2i}{32}g^2 + \frac{i}{128}gG^2 & a_{44}^{3\prime} + \frac{3i}{32}g^2 + \frac{i}{16}g'G + \frac{i}{128}gG^2 \end{matrix}\right). \tag{5.140}$$

Let $G_2(x) = \int_0^x g^2(t)dt$. Then it follows from (5.136) and (5.140) that

$$\left.\begin{matrix} a_{11}^3 = \frac{1}{384}G^3 + \frac{1}{32}G_2 + \frac{1}{16}gG & a_{22}^3 = \frac{i}{384}G^3 + \frac{i}{32}G_2 + \frac{i}{16}gG \\ a_{33}^3 = -\frac{1}{384}G^3 - \frac{1}{32}G_2 - \frac{1}{16}gG & a_{44}^3 = -\frac{i}{384}G^3 - \frac{i}{32}G_2 - \frac{i}{16}gG \end{matrix}\right\} \tag{5.141}$$

It results from (5.136), (5.135) and (5.141) that

$$\begin{cases} a_{12}^4 = \frac{1+i}{16}g'' - \frac{i}{32}g'G - \frac{1+3i}{64}g^2 - \frac{1+i}{256}gG^2 \\ a_{13}^4 = \frac{1}{32}g'' - \frac{1}{64}g'G - \frac{1}{32}g^2 - \frac{1}{256}gG^2 \\ a_{14}^4 = \frac{1-i}{16}g'' + \frac{i}{32}g'G - \frac{1-3i}{64}g^2 - \frac{1-i}{256}gG^2 \end{cases}, \tag{5.142}$$

$$\begin{cases} a_{21}^4 = \frac{1-i}{16}g'' + \frac{i}{32}g'G - \frac{1-3i}{64}g^2 - \frac{1-i}{256}gG^2 \\ a_{23}^4 = \frac{1+i}{16}g'' - \frac{i}{32}g'G - \frac{1+3i}{64}g^2 - \frac{1+i}{256}gG^2 \\ a_{24}^4 = \frac{1}{32}g'' - \frac{1}{64}g'G - \frac{1}{32}g^2 - \frac{1}{256}gG^2 \end{cases}, \tag{5.143}$$

$$\begin{cases} a_{31}^4 = \frac{1}{32}g'' - \frac{1}{64}g'G - \frac{1}{32}g^2 - \frac{1}{256}gG^2 \\ a_{32}^4 = \frac{1-i}{16}g'' + \frac{i}{32}g'G - \frac{1-3i}{64}g^2 - \frac{1-i}{256}gG^2 \\ a_{34}^4 = \frac{1+i}{16}g'' - \frac{i}{32}g'G - \frac{1+3i}{64}g^2 - \frac{1+i}{256}gG^2 \end{cases} \tag{5.144}$$

and

$$\begin{cases} a_{41}^4 = \frac{1+i}{16}g'' - \frac{i}{32}g'G - \frac{1+3i}{64}g^2 - \frac{1+i}{256}gG^2 \\ a_{42}^4 = \frac{1}{32}g'' - \frac{1}{64}g'G - \frac{1}{32}g^2 - \frac{1}{256}gG^2 \\ a_{43}^4 = \frac{1-i}{16}g'' + \frac{i}{32}g'G - \frac{1-3i}{64}g^2 - \frac{1-i}{256}gG^2 \end{cases}. \tag{5.145}$$

Let $\widehat{K}_1 = \Omega\varepsilon\varepsilon^\top\Omega^{-2}Q^{[3]}$, $\widehat{K}_2 = \Omega\varepsilon\varepsilon^\top\Omega^{-3}Q^{[2]}$, $\widehat{K} = g\Omega\varepsilon\varepsilon^\top\Omega^{-2}Q^{[3]} + g'\Omega\varepsilon\varepsilon^\top\Omega^{-3}Q^{[2]}$ and set

$$\widehat{K} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix}.$$

It follows from (5.77) that we need only the diagonal elements of the matrix $\widehat{K}$. Let $\alpha_{11}$, $\alpha_{22}$, $\alpha_{33}$, $\alpha_{44}$, be the diagonal elements of $\widehat{K}_1$ and $\beta_{11}$, $\beta_{22}$, $\beta_{33}$, $\beta_{44}$ the diagonal elements of $\widehat{K}_2$. It can be inferred from (5.119), (5.133) and (5.141) that

$$\widehat{K}_1 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ i & -i & i & -i \\ -1 & 1 & -1 & 1 \\ -i & i & -i & i \end{pmatrix}$$
$$\begin{pmatrix} a_{11}^3 & \frac{1}{8}g' + \frac{1-i}{32}gG & \frac{1}{16}g' + \frac{1}{32}gG & \frac{1}{8}g' + \frac{1+i}{32}gG \\ \frac{i}{8}g' - \frac{1-i}{32}gG & a_{22}^3 & \frac{i}{8}g' + \frac{1+i}{32}gG & \frac{i}{16}g' + \frac{i}{32}gG \\ -\frac{1}{16}g' - \frac{1}{32}gG & -\frac{1}{8}g' - \frac{1+i}{32}gG & a_{33}^3 & -\frac{1}{8}g' - \frac{1-i}{32}gG \\ -\frac{i}{8}g' - \frac{1+i}{32}gG & -\frac{i}{16}g' - \frac{i}{32}gG & -\frac{i}{8}g' + \frac{1-i}{32}gG & a_{44}^3 \end{pmatrix},$$

thus

$$\begin{cases} \alpha_{11} = \frac{1}{384}G^3 + \frac{1}{32}G_2 + \frac{3}{32}gG - \frac{1}{16}g' \\ \alpha_{22} = \frac{1}{384}G^3 + \frac{1}{32}G_2 + \frac{3}{32}gG - \frac{1}{16}g' \\ \alpha_{33} = \frac{1}{384}G^3 + \frac{1}{32}G_2 + \frac{3}{32}gG - \frac{1}{16}g' \\ \alpha_{44} = \frac{1}{384}G^3 + \frac{1}{32}G_2 + \frac{3}{32}gG - \frac{1}{16}g' \end{cases}. \tag{5.146}$$

It results from (5.127) and (5.134) that

$$\widehat{K}_2 = \begin{pmatrix} 1 & i & -1 & -i \\ i & -1 & -i & 1 \\ -1 & -i & 1 & i \\ -i & 1 & i & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{32}G^2 + \frac{1}{4}g & \frac{1+i}{8}g & -\frac{1}{8}g & \frac{1-i}{8}g \\ \frac{-1+i}{8}g & -\frac{1}{32}G^2 - \frac{1}{4}g & -\frac{1+i}{8}g & \frac{1}{8}g \\ -\frac{1}{8}g & \frac{1-i}{8}g & \frac{1}{32}G^2 + \frac{1}{4}g & \frac{1+i}{8}g \\ -\frac{1+i}{8}g & \frac{1}{8}g & \frac{-1+i}{8}g & -\frac{1}{32}G^2 - \frac{1}{4}g \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{32}G^2 + \frac{1}{8}g & -\frac{i}{32}G^2 - \frac{i}{8}g & -\frac{1}{32}G^2 - \frac{1}{8}g & \frac{i}{32}G^2 + \frac{i}{8}g \\ \frac{i}{32}G^2 + \frac{i}{8}g & \frac{1}{32}G^2 + \frac{1}{8}g & -\frac{i}{32}G^2 - \frac{i}{8}g & -\frac{1}{32}G^2 - \frac{1}{8}g \\ -\frac{1}{32}G^2 - \frac{1}{8}g & \frac{i}{32}G^2 + \frac{i}{8}g & \frac{1}{32}G^2 + \frac{1}{8}g & -\frac{i}{32}G^2 - \frac{i}{8}g \\ -\frac{i}{32}G^2 - \frac{i}{8}g & -\frac{1}{32}G^2 - \frac{1}{8}g & \frac{i}{32}G^2 + \frac{i}{8}g & \frac{1}{32}G^2 + \frac{1}{8}g \end{pmatrix}. \tag{5.147}$$

Hence

$$\begin{cases} \beta_{11} = \frac{1}{32}G^2 + \frac{1}{8}g \\ \beta_{22} = \frac{1}{32}G^2 + \frac{1}{8}g \\ \beta_{33} = \frac{1}{32}G^2 + \frac{1}{8}g \\ \beta_{44} = \frac{1}{32}G^2 + \frac{1}{8}g \end{cases}. \tag{5.148}$$

It follows from (5.146) and (5.148) that

$$\begin{cases} k_{11} = \frac{1}{384}gG^3 + \frac{1}{32}gG_2 + \frac{3}{32}g^2G + \frac{1}{16}g'g + \frac{1}{32}g'G^2 \\ k_{22} = \frac{1}{384}gG^3 + \frac{1}{32}gG_2 + \frac{3}{32}g^2G + \frac{1}{16}g'g + \frac{1}{32}g'G^2 \\ k_{33} = \frac{1}{384}gG^3 + \frac{1}{32}gG_2 + \frac{3}{32}g^2G + \frac{1}{16}g'g + \frac{1}{32}g'G^2 \\ k_{44} = \frac{1}{384}gG^3 + \frac{1}{32}gG_2 + \frac{3}{32}g^2G + \frac{1}{16}g'g + \frac{1}{32}g'G^2 \end{cases}. \tag{5.149}$$

However (5.77) and (5.149) yields

$$\begin{cases} a_{11}^{4'} = \frac{1}{1536}gG^3 + \frac{1}{128}gG_2 + \frac{3}{128}g^2G + \frac{1}{64}g'g + \frac{1}{128}g'G^2 \\ a_{22}^{4'} = \frac{1}{1536}gG^3 + \frac{1}{128}gG_2 + \frac{3}{128}g^2G + \frac{1}{64}g'g + \frac{1}{128}g'G^2 \\ a_{33}^{4'} = \frac{1}{1536}gG^3 + \frac{1}{128}gG_2 + \frac{3}{128}g^2G + \frac{1}{64}g'g + \frac{1}{128}g'G^2 \\ a_{44}^{4'} = \frac{1}{1536}gG^3 + \frac{1}{128}gG_2 + \frac{3}{128}g^2G + \frac{1}{64}g'g + \frac{1}{128}g'G^2 \end{cases}. \tag{5.150}$$

Let $\widetilde{G}(x)=\int_0^x g(t)G_2(t)dt$ and $\widehat{G}(t)=\int_0^x g'(t)G^2(t)dt$. Then it follows from (5.150) that

$$\begin{cases} a_{11}^4 = \frac{1}{6144}G^4+\frac{1}{128}\widetilde{G}+\frac{1}{128}g^2+\frac{3}{256}gG^2-\frac{1}{256}\widehat{G} \\ a_{22}^4 = \frac{1}{6144}G^4+\frac{1}{128}\widetilde{G}+\frac{1}{128}g^2+\frac{3}{256}gG^2-\frac{1}{256}\widehat{G} \\ a_{33}^4 = \frac{1}{6144}G^4+\frac{1}{128}\widetilde{G}+\frac{1}{128}g^2+\frac{3}{256}gG^2-\frac{1}{256}\widehat{G} \\ a_{44}^4 = \frac{1}{6144}G^4+\frac{1}{128}\widetilde{G}+\frac{1}{128}g^2+\frac{3}{256}gG^2-\frac{1}{256}\widehat{G} \end{cases}. \tag{5.151}$$

Set

$$\widetilde{Q}(\cdot,\mu) := \begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ q_{41} & q_{42} & q_{43} & q_{44} \end{pmatrix}.$$

It follows from (5.78) that

$$\widetilde{Q}(\cdot,\mu) = I_4+\mu^{-1}Q^{[1]}+\mu^{-2}Q^{[2]}+\mu^{-3}Q^{[3]}+\mu^{-4}Q^{[4]}+\{o(\mu^{-4})\}_\infty. \tag{5.152}$$

Thus

$$\widetilde{Q}(\cdot,\mu) = (q_{ij})_{i,j=1}^4 = I_4+(\mu^{-1}a_{ij}^1+\mu^{-2}a_{ij}^2+\mu^{-3}a_{ij}^3+\mu^{-4}a_{ij}^4)_{i,j=1}^4+\{o(\mu^{-4})\}_\infty, \tag{5.153}$$

where $a_{ij}^1$, $a_{ij}^2$, $a_{ij}^3$ and $a_{ij}^4$, for $i,j=1,2,3,4$ are given in (5.123), (5.134), (5.133), (5.141), (5.142), (5.143), (5.144), (5.145) and (5.151). We recall that $G(x)=\int_0^x g(t)dt$ and $G_2(x)=\int_0^x g^2(t)dt$. Hence

$$\begin{aligned} q_{11} &= 1+\frac{G}{4}\mu^{-1}+\left(\frac{1}{32}G^2+\frac{1}{4}g\right)\mu^{-2}+\left(\frac{1}{384}G^3+\frac{1}{32}G_2+\frac{1}{16}gG\right)\mu^{-3} \\ &\quad+\left(\frac{1}{6144}G^4+\frac{1}{128}\widetilde{G}+\frac{1}{128}g^2+\frac{3}{256}gG^2-\frac{1}{256}\widehat{G}\right)\mu^{-4}+\{o(\mu^{-4})\}_\infty \end{aligned} \tag{5.154}$$

$$\begin{aligned} q_{12} &= \frac{1+i}{8}g\mu^{-2}+\left(\frac{1}{8}g'+\frac{1-i}{32}gG\right)\mu^{-3}+\left(\frac{1+i}{16}g''-\frac{i}{32}g'G\right. \\ &\quad\left.-\frac{1+3i}{64}g^2-\frac{1+i}{256}gG^2\right)\mu^{-4}+\{o(\mu^{-4})\}_\infty \end{aligned} \tag{5.155}$$

$$\begin{aligned} q_{13} &= -\frac{1}{8}g\mu^{-2}+\left(\frac{1}{16}g'+\frac{1}{32}gG\right)\mu^{-3}+\left(\frac{1}{32}g''-\frac{1}{64}g'G-\frac{1}{32}g^2\right. \\ &\quad\left.-\frac{1}{256}gG^2\right)\mu^{-4}+\{o(\mu^{-4})\}_\infty \end{aligned} \tag{5.156}$$

$$\begin{aligned} q_{14} &= \frac{1-i}{8}g\mu^{-2}+\left(\frac{1}{8}g'+\frac{1+i}{32}gG\right)\mu^{-3}+\left(\frac{1-i}{16}g''+\frac{i}{32}g'G\right. \\ &\quad\left.-\frac{1-3i}{64}g^2-\frac{1-i}{256}gG^2\right)\mu^{-4}+\{o(\mu^{-4})\}_\infty \end{aligned} \tag{5.157}$$

$$q_{21} = -\frac{1-i}{8} g\mu^{-2} + \left(\frac{i}{8} g' - \frac{1-i}{32} gG\right)\mu^{-3} + \left(\frac{1-i}{16} g'' + \frac{i}{32} g'G \right.$$
$$\left. - \frac{1-3i}{64} g^2 - \frac{1-i}{256} gG^2\right)\mu^{-4} + \{o(\mu^{-4})\}_\infty \tag{5.158}$$

$$q_{22} = 1 - \frac{i}{4} G\mu^{-1} - \left(\frac{1}{32} G^2 + \frac{1}{4} g\right)\mu^{-2} + \left(\frac{i}{384} G^3 + \frac{i}{32} G_2 + \frac{i}{16} gG\right)\mu^{-3}$$
$$+ \left(\frac{1}{6144} G^4 + \frac{1}{128}\widetilde{G} + \frac{1}{128} g^2 + \frac{3}{256} gG^2 - \frac{1}{256}\widehat{G}\right)\mu^{-4} + \{o(\mu^{-4})\}_\infty \tag{5.159}$$

$$q_{23} = -\frac{1+i}{8} g\mu^{-2} + \left(\frac{i}{8} g' + \frac{1+i}{32} gG\right)\mu^{-3} + \left(\frac{1+i}{16} g'' - \frac{i}{32} g'G \right.$$
$$\left. - \frac{1+3i}{64} g^2 - \frac{1+i}{256} gG^2\right)\mu^{-4} + \{o(\mu^{-4})\}_\infty \tag{5.160}$$

$$q_{24} = \frac{1}{8} g\mu^{-2} + \left(\frac{i}{16} g' + \frac{i}{32} gG\right)\mu^{-3} + \left(\frac{1}{32} g'' - \frac{1}{64} g'G - \frac{1}{32} g^2 \right.$$
$$\left. - \frac{1}{256} gG^2\right)\mu^{-4} + \{o(\mu^{-4})\}_\infty \tag{5.161}$$

$$q_{31} = -\frac{1}{8} g\mu^{-2} - \left(\frac{1}{16} g' + \frac{1}{32} gG\right)\mu^{-3} + \left(\frac{1}{32} g'' - \frac{1}{64} g'G - \frac{1}{32} g^2 - \frac{1}{256} gG^2\right)\mu^{-4}$$
$$+ \{o(\mu^{-4})\}_\infty \tag{5.162}$$

$$q_{32} = \frac{1-i}{8} g\mu^{-2} - \left(\frac{1}{8} g' + \frac{1+i}{32} gG\right)\mu^{-3} + \left(\frac{1-i}{16} g'' + \frac{i}{32} g'G - \frac{1-3i}{64} g^2 \right.$$
$$\left. - \frac{1-i}{256} gG^2\right)\mu^{-4} + \{o(\mu^{-4})\}_\infty \tag{5.163}$$

$$q_{33} = 1 - \frac{1}{4} G\mu^{-1} + \left(\frac{1}{32} G^2 + \frac{1}{4} g\right)\mu^{-2} - \left(\frac{1}{384} G^3 + \frac{1}{32} G_2 + \frac{1}{16} gG\right)\mu^{-3}$$
$$+ \left(\frac{1}{6144} G^4 + \frac{1}{128}\widetilde{G} + \frac{1}{128} g^2 + \frac{3}{256} gG^2 - \frac{1}{256}\widehat{G}\right)\mu^{-4} + \{o(\mu^{-4})\}_\infty \tag{5.164}$$

$$q_{34} = \frac{1+i}{8} g\mu^{-2} - \left(\frac{1}{8} g' + \frac{1-i}{32} gG\right)\mu^{-3} + \left(\frac{1+i}{16} g'' - \frac{i}{32} g'G - \frac{1+3i}{64} g^2 \right.$$
$$\left. - \frac{1+i}{256} gG^2\right)\mu^{-4} + \{o(\mu^{-4})\}_\infty \tag{5.165}$$

$$q_{41} = -\frac{1+i}{8}g\mu^{-2} - \left(\frac{i}{8}g' + \frac{1+i}{32}gG\right)\mu^{-3} + \left(\frac{1+i}{16}g'' - \frac{i}{32}g'G - \frac{1+3i}{64}g^2 - \frac{1+i}{256}gG^2\right)\mu^{-4} + \{o(\mu^{-4})\}_\infty \tag{5.166}$$

$$q_{42} = \frac{1}{8}g\mu^{-2} - \left(\frac{i}{16}g' + \frac{i}{32}gG\right)\mu^{-3} + \left(\frac{1}{32}g'' - \frac{1}{64}g'G - \frac{1}{32}g^2 - \frac{1}{256}gG^2\right)\mu^{-4} + \{o(\mu^{-4})\}_\infty \tag{5.167}$$

$$q_{43} = -\frac{1-i}{8}g\mu^{-2} + \left(-\frac{i}{8}g' + \frac{1-i}{32}gG\right)\mu^{-3} + \left(\frac{1-i}{16}g'' + \frac{i}{32}g'G - \frac{1-3i}{64}g^2 - \frac{1-i}{256}gG^2\right)\mu^{-4} + \{o(\mu^{-4})\}_\infty \tag{5.168}$$

$$q_{44} = 1 + \frac{i}{4}G\mu^{-1} - \left(\frac{1}{32}G^2 + \frac{1}{4}g\right)\mu^{-2} - \left(\frac{i}{384}G^3 + \frac{i}{32}G_2 + \frac{i}{16}gG\right)\mu^{-3} + \left(\frac{1}{6144}G^4 + \frac{1}{128}\widetilde{G} + \frac{1}{128}g^2 + \frac{3}{256}gG^2 - \frac{1}{256}\widehat{G}\right)\mu^{-4} + \{o(\mu^{-4})\}_\infty \tag{5.169}$$

It follows from (5.65) and (5.113) that

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}, \tag{5.170}$$

from (5.64) that

$$\Xi_4 = \operatorname{diag}(1, \mu, \mu^2, \mu^3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu^2 & 0 \\ 0 & 0 & 0 & \mu^3 \end{pmatrix} \tag{5.171}$$

and from (5.66) that

$$E(\cdot, \mu) = \operatorname{diag}\left(e^{\mu x}, e^{i\mu x}, e^{-\mu x}, e^{-i\mu x}\right) = \begin{pmatrix} e^{\mu x} & 0 & 0 & 0 \\ 0 & e^{i\mu x} & 0 & 0 \\ 0 & 0 & e^{-\mu x} & 0 \\ 0 & 0 & 0 & e^{-i\mu x} \end{pmatrix}. \tag{5.172}$$

Let

$$Y(\cdot,\mu) := \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & y_{22} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{33} & y_{34} \\ y_{41} & y_{42} & y_{43} & y_{44} \end{pmatrix}.$$

We recall that $G(x) = \int_0^x g(t)dt$, $G_2(x) = \int_0^x g^2(t)dt$, $\widetilde{G}(x) = \int_0^x g(t)G_2(t)dt$ and $\widehat{G}(x) = \int_0^x g'(t)G^2(t)dt$. Then it follows from (5.79) that $Y(\cdot,\mu) = \Xi_4(\mu)V\widetilde{Q}(\cdot,\mu)E(\cdot,\mu)$. Thus

$$\begin{cases} y_{11} = (q_{11}+q_{21}+q_{31}+q_{41})e^{\mu x} \\ y_{21} = (\mu q_{11}+i\mu q_{21}-\mu q_{31}-i\mu q_{41})e^{\mu x} \\ y_{31} = (\mu^2 q_{11}-\mu^2 q_{21}+\mu^2 q_{31}-\mu^2 q_{41})e^{\mu x} \\ y_{41} = (\mu^3 q_{11}-i\mu^3 q_{21}-\mu^3 q_{31}+i\mu^3 q_{41})e^{\mu x} \end{cases}, \tag{5.173}$$

$$\begin{cases} y_{12} = (q_{12}+q_{22}+q_{32}+q_{42})e^{i\mu x} \\ y_{22} = (\mu q_{12}+i\mu q_{22}-\mu q_{32}-i\mu q_{42})e^{i\mu x} \\ y_{32} = (\mu^2 q_{12}-\mu^2 q_{22}+\mu^2 q_{32}-\mu^2 q_{42})e^{i\mu x} \\ y_{42} = (\mu^3 q_{12}-i\mu^3 q_{22}-\mu^3 q_{32}+i\mu^3 q_{42})e^{i\mu x} \end{cases}, \tag{5.174}$$

$$\begin{cases} y_{13} = (q_{13}+q_{23}+q_{33}+q_{43})e^{-\mu x} \\ y_{23} = (\mu q_{13}+i\mu q_{23}-\mu q_{33}-i\mu q_{43})e^{-\mu x} \\ y_{33} = (\mu^2 q_{13}-\mu^2 q_{23}+\mu^2 q_{33}-\mu^2 q_{43})e^{-\mu x} \\ y_{43} = (\mu^3 q_{13}-i\mu^3 q_{23}-\mu^3 q_{33}+i\mu^3 q_{43})e^{-\mu x} \end{cases} \tag{5.175}$$

and

$$\begin{cases} y_{14} = (q_{14}+q_{24}+q_{34}+q_{44})e^{-i\mu x} \\ y_{24} = (\mu q_{14}+i\mu q_{24}-\mu q_{34}-i\mu q_{44})e^{-i\mu x} \\ y_{34} = (\mu^2 q_{14}-\mu^2 q_{24}+\mu^2 q_{34}-\mu^2 q_{44})e^{-i\mu x} \\ y_{44} = (\mu^3 q_{14}-i\mu^3 q_{24}-\mu^3 q_{34}+i\mu^3 q_{44})e^{-i\mu x} \end{cases}. \tag{5.176}$$

**Proposition 5.32.** *We require that $g \in C^2[a,b]$. Let $G(x) = \int_0^x g(t)dt$, $G_2(x) = \int_0^x g^2(t)dt$, $\widetilde{G}(x) = \int_0^x g(t)G_2(t)dt$ and $\widehat{G}(x) = \int_0^x g'(t)G^2(t)dt$. Let $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ be a fundamental system of $y^{(4)} - (gy')' = \mu^4 y$. Then it follows from* (5.153) *and* (5.173)–(5.176) *that*

$$\begin{aligned}\eta_1 =& y_{11} = \left(1 + \frac{G}{4}\mu^{-1} + \left(\frac{1}{32}G^2 - \frac{1}{8}g\right)\mu^{-2} + \left(\frac{1}{384}G^3 + \frac{1}{32}G_2 - \frac{1}{32}gG - \frac{1}{16}g'\right)\mu^{-3}\right.\\ &\left. + \left(\frac{1}{6144}G^4 + \frac{1}{128}\widetilde{G} - \frac{7}{128}g^2 - \frac{1}{256}\widehat{G} + \frac{5}{32}g'' - \frac{1}{64}g'G\right)\mu^{-4}\right)e^{\mu x}\\ & + \{o(\mu^{-4})\}_\infty e^{\mu x}\end{aligned} \tag{5.177}$$

$$\begin{aligned}\eta_2 =& y_{12} = \left(1 - \frac{i}{4}G\mu^{-1} - \left(\frac{1}{32}G^2 - \frac{1}{8}g\right)\mu^{-2} + \left(\frac{i}{384}G^3 + \frac{i}{32}G_2 - \frac{i}{32}gG - \frac{i}{16}g'\right)\mu^{-3}\right.\\ &\left. + \left(\frac{1}{6144}G^4 + \frac{1}{128}\widetilde{G} - \frac{7}{128}g^2 - \frac{1}{256}\widehat{G} + \frac{5}{32}g'' - \frac{1}{64}g'G\right)\mu^{-4}\right)e^{i\mu x}\\ & + \{o(\mu^{-4})\}_\infty e^{i\mu x}\end{aligned} \tag{5.178}$$

$$\begin{aligned}\eta_3 =& y_{13} = \left(1 - \frac{1}{4}G\mu^{-1} + \left(\frac{1}{32}G^2 - \frac{1}{8}g\right)\mu^{-2} - \left(\frac{1}{384}G^3 + \frac{1}{32}G_2 - \frac{1}{32}gG - \frac{1}{16}g'\right)\mu^{-3}\right.\\ &\left. + \left(\frac{1}{6144}G^4 + \frac{1}{128}\widetilde{G} - \frac{7}{128}g^2 - \frac{1}{256}\widehat{G} + \frac{5}{32}g'' - \frac{1}{64}g'G\right)\mu^{-4}\right)e^{-\mu x}\\ & + \{o(\mu^{-4})\}_\infty e^{-\mu x}\end{aligned} \tag{5.179}$$

$$\begin{aligned}\eta_4 =& y_{14} = \left(1 + \frac{i}{4}G\mu^{-1} - \left(\frac{1}{32}G^2 - \frac{1}{8}g\right)\mu^{-2} - \left(\frac{i}{384}G^3 + \frac{i}{32}G_2 - \frac{i}{32}gG - \frac{i}{16}g'\right)\mu^{-3}\right.\\ &\left. + \left(\frac{1}{6144}G^4 + \frac{1}{128}\widetilde{G} - \frac{7}{128}g^2 - \frac{1}{256}\widehat{G} + \frac{5}{32}g'' - \frac{1}{64}g'G\right)\mu^{-4}\right)e^{-i\mu x}\\ & + \{o(\mu^{-4})\}_\infty e^{-i\mu x}.\end{aligned} \tag{5.180}$$

**Corollary 5.33.** *We require that $g \in C^2[a,b]$. Let $G(x) = \int_0^x g(t)dt$, $G_2(x) = \int_0^x g^2(t)dt$, $\widetilde{G}(x) = \int_0^x g(t)G_2(t)dt$ and $\widehat{G}(x) = \int_0^x g'(t)G^2(t)dt$. Then the differential equation $y^{(4)} - (gy')' = \mu^4 y$ has an asymptotic fundamental system $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ of the form*

$$\eta_\nu = \sum_{r=0}^{4} (\mu i^{\nu-1})^{-r}\varphi_r(x)e^{\mu i^{\nu-1}x} + o(\mu^{-4})e^{\mu i^{\nu-1}x}, \tag{5.181}$$

*where*

$$\begin{cases} \varphi_0(x) = 1, \\ \varphi_1(x) = \frac{1}{4}G(x), \\ \varphi_2(x) = \frac{1}{32}G^2(x) - \frac{1}{8}g(x), \\ \varphi_3(x) = \frac{1}{384}G^3(x) + \frac{1}{32}G_2(x) - \frac{1}{32}(gG)(x) - \frac{1}{16}g'(x), \\ \varphi_4(x) = \frac{1}{6144}G^4(x) + \frac{1}{128}\widetilde{G}(x) - \frac{7}{128}g^2(x) - \frac{1}{256}\widehat{G}(x) + \frac{5}{32}g''(x) - \frac{1}{64}(g'G)(x). \end{cases} \tag{5.182}$$

**Remark 5.34.** It follows from Proposition 5.32 that a fundamental system of $y^{(4)} - (gy')' = \mu^4 y$ is $\{\eta_1, \eta_2, \eta_3, \eta_4\}$, where

$$\begin{cases} \eta_1 &= (1 + O(\mu^{-1}))e^{\mu x}, \\ \eta_2 &= (1 + O(\mu^{-1}))e^{i\mu x,} \\ \eta_3 &= (1 + O(\mu^{-1}))e^{-\mu x}, \\ \eta_4 &= (1 + O(\mu^{-1}))e^{-i\mu x}. \end{cases} \tag{5.183}$$

Thus

$$\left\{(1 + O(\mu^{-1}))\cos(\mu x), (1 + O(\mu^{-1}))\sin(\mu x), (1 + O(\mu^{-1}))\cosh(\mu x), (1 + O(\mu^{-1}))\sinh(\mu x)\right\}$$

is a fundamental system of $y^{(4)} - (gy')' = \mu^4 y$.

**Note 5.35.** In the remainder of the chapter, we consider the asymptotic fundamental system $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ of the form

$$\eta_\nu = \sum_{r=0}^{2} (\mu i^{\nu-1})^{-r} \varphi_r(x) e^{\mu i^{\nu-1} x} + o(\mu^{-2}) e^{\mu i^{\nu-1} x}, \tag{5.184}$$

$$\delta_{\nu,j} = \left\{ \left[\frac{d^j}{dx^j}\right] \left( \sum_{r=0}^{2} (\mu i^{\nu-1})^{-r} \varphi_r(x) e^{\mu i^{\nu-1} x} \right) \right\} e^{-i^{\nu-1}\mu x} + o(\mu^{-2+j}), \tag{5.185}$$

where $\nu = 1, 2, 3, 4$ and $\varphi_0$, $\varphi_1$ and $\varphi_2$ are as defined in (5.182). We investigate the asymptotics of the eigenvalue of the problem (3.1)–(3.2) with $g \neq 0$. We consider different cases of the boundary terms $B_1 y$ and $B_2 y$ where $B_1 y$ and $B_2(y)$ are the following: $B_1 y = y^{[p_1]}(0)$ and $B_2 y = y^{[p_2]}(0)$, with $p_1 < p_2$ and $p_1 + p_2 \neq 3$. Using (5.185) and the set of the boundary conditions $B_j(\mu^2) y = 0$, $j = 1, 2, 3, 4$, see (3.2), we denote

$$\alpha_{j,k} = \gamma_{j,k} \exp(\varepsilon_{j,k}), \quad j, k = 1, 2, 3, 4, \tag{5.186}$$

where

$$\begin{cases} \varepsilon_{1,k} = \varepsilon_{2,k} = 0,\ \varepsilon_{3,k} = \varepsilon_{4,k} = i^{k-1}\mu a, \\ \gamma_{1,k} = \delta_{k,p_1}(0,\mu),\ \gamma_{2,k} = \delta_{k,p_2}(0,\mu) \text{ if } p_2 \le 2, \\ \gamma_{2,k} = \delta_{k,3}(0,\mu) - g(0)\delta_{k,1}(0,\mu) \text{ if } p_2 = 3, \\ \gamma_{3,k} = \delta_{k,p_3}(a,\mu) + i\alpha\mu^2\delta_{k,3-p_3}(a,\mu) \text{ if } p_3 = 2, \\ \gamma_{3,k} = \delta_{k,p_3}(a,\mu) - i\alpha\mu^2\delta_{k,3-p_3}(a,\mu) \text{ if } p_3 = 1, \\ \gamma_{4,k} = \delta_{k,0}(a,\mu) + i\alpha\mu^2(\delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu)) \text{ if } p_4 = 0, \\ \gamma_{4,k} = \delta_{k,p_4}(a,\mu) - g(a)\delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,3-p_4}(a,\mu) \text{ if } p_4 = 3. \end{cases} \tag{5.187}$$

It follows from (5.185) that

$$\delta_{\nu,0}(x,\mu) = 1 + \frac{1}{4}G(x)i^{1-\nu}\mu^{-1} + \Big(\frac{1}{32}G^2(x) - \frac{1}{8}g(x)\Big)i^{2(1-\nu)}\mu^{-2} + o(\mu^{-2}) \tag{5.188}$$

$$\delta_{\nu,1}(x,\mu) = \mu i^{\nu-1} + \frac{1}{4}G(x) + \Big(\frac{1}{32}G^2(x) + \frac{1}{8}g(x)\Big)i^{1-\nu}\mu^{-1} + o(\mu^{-1}) \tag{5.189}$$

$$\delta_{\nu,2}(x,\mu) = \mu^2 i^{2(\nu-1)} + \frac{1}{4}G(x)\mu i^{\nu-1} + \frac{1}{32}G^2(x) + \frac{3}{8}g(x) + \frac{1}{8}(gG)(x)i^{1-\nu} + o(1) \tag{5.190}$$

$$\delta_{\nu,3}(x,\mu) = \mu^3 i^{3(\nu-1)} + \frac{1}{4}G(x)\mu^2 i^{2(\nu-1)} + \Big(\frac{1}{32}G^2(x) + \frac{5}{8}g(x)\Big)i^{\nu-1}\mu + o(\mu). \tag{5.191}$$

It follows from (5.186), that the characteristic function for general $g$, (the equivalent for general $g$ of the equation (4.20) derived for $g = 0$), of the problem (3.1)–(3.2) is

$$D(\mu) = \det(\gamma_{j,k}\exp(\varepsilon_{j,k}))_{j,k=1}^4. \tag{5.192}$$

Then it follows from (5.192) that

$$D(\mu) = \sum_{m=1}^{5} \psi_m(\mu)e^{\omega_m\mu a}, \tag{5.193}$$

where

$$\omega_1 = 1 + i,\ \omega_2 = -1 + i,\ \omega_3 = -1 - i,\ \omega_4 = 1 - i,\ \omega_5 = 0. \tag{5.194}$$

Therefore

$$D_1(\mu) := D(\mu)e^{-\omega_1\mu a} = \psi_1 + \sum_{m=2}^{5} \psi_m(\mu)e^{(\omega_m - \omega_1)\mu a}, \tag{5.195}$$

where $\omega_2 - \omega_1 = -2$, $\omega_3 - \omega_1 = -2 - 2i$, $\omega_4 - \omega_1 = -2i$, $\omega_5 - \omega_1 = -1 - i$.

It follows that for $\arg\mu \in [-\frac{3\pi}{8}, \frac{\pi}{8}]$, we have $|e^{(\omega_m-\omega_1)\mu a}| \le e^{-\sin\frac{\pi}{8}|\mu|a}$ for $m = 2, 3, 5$ and these terms can be absorbed by $\psi_1(\mu)$ as there are of the form $o(\mu^{-s})$ for any integer $s$. Thus for $\arg\mu \in [-\frac{3\pi}{8}, \frac{\pi}{8}]$,

$$D_1(\mu) = \psi_1(\mu) + \psi_4(\mu)e^{(\omega_4-\omega_1)\mu a} = \psi_1(\mu) + \psi_4(\mu)e^{-2i\mu a}, \tag{5.196}$$

where

$$\psi_1(\mu) = [\gamma_{1,3}\gamma_{2,4} - \gamma_{2,3}\gamma_{1,4}][\gamma_{3,1}\gamma_{4,2} - \gamma_{3,2}\gamma_{4,1}], \tag{5.197}$$

$$\psi_4(\mu) = [\gamma_{1,2}\gamma_{2,3} - \gamma_{2,2}\gamma_{1,3}][\gamma_{3,1}\gamma_{4,4} - \gamma_{3,4}\gamma_{4,1}]. \tag{5.198}$$

**Remark 5.36.** We already know for each of the following sections the asymptotic distribution of the eigenvalues for $g = 0$. Denote by $D_0$ the corresponding characteristic function $D$. The Birkhoff regularity, see Proposition 5.31, implies that $\psi_1$ and $\psi_4$ have nonzero terms apart from $o$-terms. The characteristic function of the differential equation (3.1) as defined in (2.36) is $\pi(\rho) = \rho^{(4)} - 1$. This characteristic function does not depend on the function $g$, then the matrix function $C_1(x)$ and therefore the matrix function $C(x,\lambda)$ defined in Theorem 5.20 do not depend on $g$. Also the coefficients of $y^{[3]}(a)$, $y''(a)$, $y'(a)$ and $y(a)$ in the boundary conditions $B_j(\lambda)y = y^{[p_j]}(a)+i\varepsilon_j\alpha\lambda y^{[q_j]}(a) = 0$, $j = 3, 4$ for general $g$ are respectively the same as the coefficients of $y^{(3)}(a)$, $y''(a)$, $y'(a)$ and $y(a)$ in the corresponding boundary conditions for $g = 0$ and the coefficients of $y^{[p_j]}(0)$ in the boundary conditions $B_j(\lambda)y = y^{[p_j]}(0) = 0$ where $j = 1, 2$ and $p_1 + p_2 \ne 3$ are respectively the same as in the corresponding boundary conditions for $g = 0$. Hence the boundary matrices $W^{(j)}(\lambda)$, as defined in (5.47), of each of the sixteen eigenvalue problems are the same for $g = 0$ and for general $g$. Due to the Birkhoff regularity, $g$ only influences lower order terms in $D$, thus higher order terms do not vanish from $D$. Therefore it can be inferred from Remark 5.34 that away from the small rectangles $R_k$, $R_{-k}$, $\widetilde{R}_k$, $\widetilde{R}_{-k}$, see (4.47), around the zeros of $D_0$, $|D(\mu) - D_0(\mu)| < |D_0(\mu)|$ if $|\mu|$ is sufficiently large. The fundamental system $y_j$, $j = 1, 2, 3, 4$, with $y_j^{[m]}(0) = \delta_{j,m+1}$ for $m = 0, 1, 2, 3$ and where $\delta$ is the Kronecker's delta, depends analytically on $\mu$, then $D$ and $D_0$ are analytic functions. Since $g$ influences only lower order terms in $D$, then the functions $D$ and $D_0$ are asymptotically close. Hence applying Rouché's theorem both to large squares $S_k$ defined in (4.66) and avoiding the small rectangles defined in (4.47) and to the boundaries of these small rectangles which are sufficiently far away from the zeros of $D_0$, it follows that the eigenvalue problem for general $g$ has the same asymptotic distribution as for $g = 0$. Hence the $o(1)$ asymptotic distribution of the eigenvalues for general $g$ is the same as the corresponding asymptotic distribution of the eigenvalues for $g = 0$. Due to the symmetry of the problem, see Proposition 3.13, we will only need to find the eigenvalue asymptotics along the positive real axis.

Let

$$\mu_k = k\frac{\pi}{a} + \tau(k),\ \tau(k) = \sum_{m=0}^{n} \tau_k k^{-m} + o(k^{-n}), \quad k = 1, 2, \dots. \tag{5.199}$$

Next we find the terms $\tau_1$ and $\tau_2$ in the expansion of $\tau(k)$. Applying the second order Taylor expansion of the functions $x \mapsto \frac{1}{1+x}$ and $x \mapsto e^{-x}$ in the neighborhood of 0 respectively to $\frac{1}{\mu_k}$ and $e^{-2i\mu_k a}$ where $k$ is large enough we get

$$\frac{1}{\mu_k} = \frac{a}{k\pi}\Big(1 + \frac{a\tau(k)}{k\pi}\Big)^{-1} = \frac{a}{k\pi} - \frac{a^2\tau_0}{k^2\pi^2} + o(k^{-2}), \tag{5.200}$$

while

$$\begin{aligned} e^{-2i\mu_k a} &= e^{-2i\tau(k)a} = e^{-2i\tau_0 a} \exp\Big( -2ia\Big(\frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2})\Big)\Big) \\ &= e^{-2i\tau_0 a}\Big(1 - 2ia\tau_1\frac{1}{k} - (2a^2\tau_1^2 + 2ia\tau_2)\frac{1}{k^2} + o(k^{-2})\Big). \end{aligned} \tag{5.201}$$

We know that $D_1(\mu_k) = 0$ can be written as

$$\mu_k^{-\gamma}\psi_1(\mu_k) + \mu_k^{-\gamma}\psi_4(\mu_k)e^{-2i\tau_k a} = 0, \tag{5.202}$$

where $\gamma$ is the highest $\mu$-power in $\psi_1(\mu)$ and $\psi_4(\mu)$.

## 5.10 The boundary terms $B_1y$ and $B_2y$ are the following: $B_1y = y(0)$ and $B_2y = y''(0)$

In this section $\gamma_{1,k} = \delta_{k,0}(0, \mu)$ and $\gamma_{2,k} = \delta_{k,2}(0, \mu)$, where $k = 1, 2, 3, 4$. We give the functions $\psi_1$ and $\psi_4$ for each of different cases of $\gamma_{3,1}$, $\gamma_{3,2}$, $\gamma_{3,4}$, $\gamma_{4,1}$, $\gamma_{4,2}$ and $\gamma_{4,4}$, where $\gamma_{3,1}$, $\gamma_{3,2}$, $\gamma_{3,4}$, $\gamma_{4,1}$, $\gamma_{4,2}$ and $\gamma_{4,4}$ are as defined in (5.187), and we investigate the corresponding asymptotic eigenvalues.

It follows from (5.187) that

$$\gamma_{1,3}\gamma_{2,4} - \gamma_{2,3}\gamma_{1,4} = -2\mu^2 + o(\mu^2), \tag{5.203}$$

$$\gamma_{1,2}\gamma_{2,3} - \gamma_{2,2}\gamma_{1,3} = 2\mu^2 + o(\mu^2). \tag{5.204}$$

### 5.10.1 Asymptotic of the eigenvalue for $B_3y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4y = y^{[3]}(a) - i\alpha\mu^2 y(a)$

Remark 5.36 and Proposition 4.23 lead to the following proposition

**Proposition 5.37.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y(0)$, $B_2(y) = y''(0)$, $B_3y = y''(a) + i\alpha\lambda y'(a)$ and $B_4y = y^{[3]}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\hat{\mu}_k^{\pm} = \pm(2k-1)\frac{\pi}{2a} + o(1), \ \ if \ \ k > 0$$
$$\hat{\mu}_k^{\pm} = \pm i(2|k|-1)\frac{\pi}{2a} + o(1), \ \ if \ \ k < 0$$

*as $|k| \to \infty$. In particular, there is an odd number of pure imaginary eigenvalues.*

In this subsection, $\gamma_{3,k} = \delta_{k,2}(a,\mu) + i\alpha\mu^2\delta_{k,1}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,0}(a,\mu)$.

We calculate explicitly $\psi_1$ and $\psi_4$ as given respectively in (5.197) and (5.198). It follows from (5.187) that

$$\begin{aligned}\gamma_{3,1}\gamma_{4,2} - \gamma_{3,2}\gamma_{4,1} &= 2\alpha\mu^6 + (1-i)(1+\alpha^2+2\alpha\varphi_1(a))\mu^5 \\ &\quad - 2i((1+\alpha^2)\varphi_1(a) + \alpha(1+\varphi_1^2(a)))\mu^4 + o(\mu^4),\end{aligned} \tag{5.205}$$

$$\begin{aligned}\gamma_{3,1}\gamma_{4,4} - \gamma_{3,4}\gamma_{4,1} &= -2\alpha\mu^6 + (1+i)(1+\alpha^2-2\alpha\varphi_1(a))\mu^5 \\ &\quad - 2i(\alpha(1+\varphi_1^2(a)) + (1-\alpha^2)\varphi_1(a))\mu^4 + o(\mu^4)\ .\end{aligned} \tag{5.206}$$

The equations (5.205) and (5.206) are respectively given by the lines "The computation of psi12" and "The computation of psi42" of Subsection 7.4.2. Putting (5.203), (5.204), (5.205) and (5.206) together, we get, from (5.197) and (5.198),

$$\psi_1(\mu) = -4\alpha\mu^8 - (1-i)(2(1+\alpha^2) + \alpha G(a))\mu^7 \\ + i\Big((1+\alpha^2)G(a) + 4\alpha + \frac{1}{4}\alpha G^2(a)\Big)\mu^6 + o(\mu^6), \tag{5.207}$$

$$\psi_4(\mu) = -4\alpha\mu^8 + (1+i)(2(1+\alpha^2) - \alpha G(a))\mu^7 \\ + i\Big((1+\alpha^2)G(a) - \frac{1}{4}\alpha G^2(a) - 4\alpha\Big)\mu^6 + o(\mu^6). \tag{5.208}$$

Recall that

$$\tau_0 = -\frac{\pi}{2a}, \tag{5.209}$$

see Proposition 5.37. Substituting (5.207), (5.208), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\tau_1 = \frac{i}{2}\frac{1+\alpha^2}{\alpha\pi} + \frac{1}{4}\frac{G(a)}{\pi} \tag{5.210}$$

$$\tau_2 = \frac{i}{4}\frac{1+\alpha^2}{\alpha\pi} - \frac{a(1-\alpha^2)^2}{4\pi^2\alpha^2} + \frac{1}{8}\frac{G(a)}{\pi}. \tag{5.211}$$

The values of $\tau_1$ and $\tau_2$ given respectively in the equations (5.210) and (5.211) are obtained from the lines of code "The computations of $\tau_1$ and $\tau_2$" of Subsection 7.4.2. Thus the below theorem follows

**Theorem 5.38.** *For $g \in C^1[0,a]$, there exists a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, counted with multiplicity, of the problem* (3.1)–(3.2)*, where $B_1(y) = y(0)$, $B_2(y) = y''(0)$, $B_3y = y''(a) + i\alpha\lambda y'(a)$ and $B_4y = y^{[3]}(a) - i\alpha\lambda y(a)$, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda}_k$ for $k \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}),$$

*where the numbers $\tau_0$, $\tau_1$ and $\tau_2$ are as respectively defined in* (5.209), (5.210) *and* (5.211)*. In particular, there is an odd number of imaginary eigenvalues.*

### 5.10.2 Asymptotic of the eigenvalue for $B_3y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4y = y(a) + i\alpha\mu^2 y^{[3]}(a)$

Remark 5.36 and Proposition 4.25 lead to the following proposition

**Proposition 5.39.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y(0)$, $B_2(y) = y''(0)$, $B_3 y = y''(a) + i\alpha\lambda y'(a)$ and $B_4 y = y(a) + i\alpha\lambda 2y^{[3]}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\hat{\mu}_k^{\pm} = \pm(2k-3)\frac{\pi}{2a} + o(1), \ \text{if} \ k > 0$$
$$\hat{\mu}_k^{\pm} = \pm i(2|k|-3)\frac{\pi}{2a} + o(1), \ \text{if} \ k < 0$$

*as $|k| \to \infty$. In particular, there is an even number of pure imaginary eigenvalues.*

In this subsection, $\gamma_{3,k} = \delta_{k,2}(a,\mu) + i\alpha\mu^2\delta_{k,1}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,0}(a,\mu) + i\alpha\mu^2(\delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu))$.

It follows from (5.187) that

$$\begin{aligned}\gamma_{3,1}\gamma_{4,2} - \gamma_{3,2}\gamma_{4,1} &= 2i\alpha^2\mu^8 + (1+i)\alpha\Big(\frac{1}{2}\alpha G(a) + 1\Big)\mu^7 \\ &\quad + \frac{1}{2}\alpha G(a)\Big(\frac{1}{4}\alpha G(a) + 1\Big)\mu^6 + o(\mu^6), \end{aligned} \tag{5.212}$$

$$\begin{aligned}\gamma_{3,1}\gamma_{4,4} - \gamma_{3,4}\gamma_{4,1} &= -2i\alpha^2\mu^8 + (1-i)\alpha\Big(\frac{1}{2}\alpha G(a) - 1\Big)\mu^7 \\ &\quad + \frac{1}{2}\alpha G(a)\Big(\frac{1}{4}\alpha G(a) - 1\Big)\mu^6 + o(\mu^6). \end{aligned} \tag{5.213}$$

Putting (5.203), (5.204), (5.212) and (5.213) together, it follows from (5.197) and (5.198) that

$$\psi_1(\mu) = -4i\alpha^2\mu^{10} - (1+i)(\alpha^2 G(a) + 2\alpha)\mu^9 - \Big(\frac{1}{4}\alpha^2 G^2(a) + \alpha G(a)\Big)\mu^8 + o(\mu^8), \tag{5.214}$$

$$\psi_4(\mu) = -4i\alpha^2\mu^{10} + (1-i)(\alpha^2 G(a) - 2\alpha)\mu^9 - \Big(\frac{1}{4}\alpha^2 G^2(a) - \alpha G(a)\Big)\mu^8 + o(\mu^8). \tag{5.215}$$

On the other hand

$$\tau_0 = -\frac{3\pi}{2a}, \tag{5.216}$$

see Proposition 5.39. Substituting (5.214), (5.215), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\tau_1 = \frac{i}{2\alpha\pi} + \frac{1}{4}\frac{G(a)}{\pi}, \tag{5.217}$$

$$\tau_2 = \frac{3i}{4\alpha\pi} - \frac{a}{4\pi^2\alpha^2} + \frac{3}{8}\frac{G(a)}{\pi}, \tag{5.218}$$

and the following theorem

**Theorem 5.40.** *For $g \in C^1[0,a]$, there exists a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, counted with multiplicity, of the problem* (3.1)–(3.2)*, where $B_1(y) = y(0)$, $B_2(y) = y''(0)$, $B_3 y = y''(a) + i\alpha\lambda y'(a)$ and $B_4 y = y(a) + i\alpha\lambda y^{[3]}(a)$, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda_k}$ for $k \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}),$$

*where the numbers $\tau_0$, $\tau_1$ and $\tau_2$ are as respectively defined in* (5.216), (5.217) *and* (5.218)*. In particular, there is an even number of imaginary eigenvalues.*

### 5.10.3 Asymptotic of the eigenvalue for $B_3 y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4 y = y^{[3]}(a) - i\alpha\mu^2 y(a)$

Remark 5.36 and Proposition 4.29 lead to the following proposition

**Proposition 5.41.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2)*, where $B_1(y) = y(0)$, $B_2(y) = y''(0)$, $B_3 y = y'(a) - i\alpha\lambda y''(a)$ and $B_4 y = y^{[3]}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\hat{\mu}_k^{\pm} = \pm(4k-3)\frac{\pi}{4a} + o(1), \ \ if \ \ k > 0$$
$$\hat{\mu}_k^{\pm} = \pm i(4|k|-3)\frac{\pi}{4a} + o(1), \ \ if \ \ k < 0$$

*as $|k| \to \infty$. In particular, there is an odd number of pure imaginary eigenvalues.*

In this subsection, $\gamma_{3,k} = \delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,2}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,0}(a,\mu)$.

It follows from (5.187) that

$$\begin{aligned}\gamma_{3,1}\gamma_{4,2}-\gamma_{3,2}\gamma_{4,1} &= -(1+i)\alpha\mu^7-\Big(\frac{1}{2}\alpha G(a)+2\alpha^2\Big)\mu^6\\ &\quad-(1-i)\alpha\Big(\frac{1}{16}G^2(a)+\frac{1}{2}\alpha G(a)+\frac{3}{4}g(a)\Big)\mu^5+o(\mu^5), \qquad (5.219)\\ \gamma_{3,1}\gamma_{4,4}-\gamma_{3,4}\gamma_{4,1} &= (1-i)\alpha\mu^7+\Big(\frac{1}{2}\alpha G(a)-2\alpha^2\Big)\mu^6\\ &\quad+(1+i)\alpha\Big(\frac{1}{16}G^2(a)-\frac{1}{2}\alpha G(a)+\frac{3}{4}g(a)\Big)\mu^5+o(\mu^5). \qquad (5.220)\end{aligned}$$

We now calculate $\psi_1$ and $\psi_4$ as given in (5.197) and (5.198). It follows from (5.203), (5.204), (5.205) and (5.206) that

$$\begin{aligned}\psi_1(\mu) &= 2(1+i)\alpha\mu^9+(\alpha G(a)+4\alpha^2)\mu^8\\ &\quad+(1-i)\alpha\Big(\frac{1}{8}G^2(a)+\alpha G(a)+\frac{3}{2}g(a)\Big)\mu^7+o(\mu^7), \qquad (5.221)\\ \psi_4(\mu) &= 2(1-i)\alpha\mu^9+(\alpha G(a)-4\alpha^2)\mu^8\\ &\quad+(1+i)\alpha\Big(\frac{1}{8}G^2(a)+\alpha G(a)+\frac{3}{2}g(a)\Big)\mu^7+o(\mu^7), \qquad (5.222)\end{aligned}$$

and from Proposition 5.41 that

$$\tau_0=-\frac{3\pi}{4a}. \qquad (5.223)$$

Putting (5.221), (5.222), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\tau_1=\frac{i\alpha}{\pi}+\frac{1}{\pi}\frac{G(a)}{4}, \qquad (5.224)$$

$$\tau_2=\frac{3i\alpha}{4\pi}-\frac{a\alpha^2}{\pi^2}+\frac{3a}{4}\frac{g(a)}{\pi^2}+\frac{3}{16}\frac{G(a)}{\pi}, \qquad (5.225)$$

and the following result

**Theorem 5.42.** *For $g\in C^1[0,a]$, there exists a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k\in\mathbb{Z}$, counted with multiplicity, of the problem* (3.1)–(3.2)*, where $B_1(y)=y(0)$, $B_2(y)=y''(0)$, $B_3y=y'(a)-i\alpha\lambda y''(a)$ and $B_4y=y^{[3]}(a)-i\alpha\lambda y(a)$, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k|<k_0$ and $\lambda_{-k}=-\overline{\lambda}_k$ for $k\geq k_0$, where $\lambda_k=\mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k=k\frac{\pi}{a}+\tau_0+\frac{\tau_1}{k}+\frac{\tau_2}{k^2}+o(k^{-2}),$$

*where the numbers $\tau_0$, $\tau_1$ and $\tau_2$ are as respectively defined in* (5.223), (5.224) *and* (5.225)*. In particular, there is an odd number of imaginary eigenvalues.*

### 5.10.4 Asymptotic of the eigenvalue for $B_3y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4y = y(a) + i\alpha\mu^2 y^{[3]}(a)$

Remark 5.36 and Proposition 4.32 lead to the following proposition

**Proposition 5.43.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y(0)$, $B_2(y) = y''(0)$, $B_3y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4y = y(a) + i\alpha\mu^2 y^{[3]}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\begin{array}{l} \hat{\mu}_k^{\pm} = \pm(4k-7)\frac{\pi}{4a} + o(1), \quad if\ k > 0 \\ \hat{\mu}_k^{\pm} = \pm i(4|k|-7)\frac{\pi}{4a} + o(1), \quad if\ k < 0 \end{array}.$$

*In particular, there is an even number of pure imaginary eigenvalues.*

The terms $\gamma_{3,k}$ and $\gamma_{4,k}$ in this subsection are as follow $\gamma_{3,k} = \delta_{k,1}(a,\mu) + i\alpha\mu^2\delta_{k,2}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,0}(a,\mu) + i\alpha\mu^2(\delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu))$.

It follows from (5.187) that

$$\begin{aligned} \gamma_{3,1}\gamma_{4,2} - \gamma_{3,2}\gamma_{4,1} &= -(1-i)\alpha^2\mu^7 - \frac{1}{2}i\alpha^2 G(a)\mu^6 \\ &\quad - (1+i)\alpha^2\Big(\frac{1}{16}G^2(a) + \frac{3}{4}g(a)\Big)\mu^5 + o(\mu^5), \qquad (5.226) \\ \gamma_{3,1}\gamma_{4,4} - \gamma_{3,4}\gamma_{4,1} &= (1+i)\alpha^2\mu^7 + \frac{1}{2}i\alpha^2 G(a)\mu^6 \\ &\quad - (1-i)\alpha^2\Big(\frac{1}{16}G^2(a) + \frac{3}{4}g(a)\Big)\mu^5 + o(\mu^5). \qquad (5.227) \end{aligned}$$

It follows from (5.203), (5.204), (5.226) and (5.227) that the functions $\psi_1$ and $\psi_4$ as given in (5.197) and (5.198) are as follow

$$\psi_1(\mu) = -2(1-i)\alpha^2\mu^9 + i\alpha^2 G(a)\mu^8 + (1+i)\alpha^2\Big(\frac{1}{8}G^2(a) + \frac{3}{2}g(a)\Big)\mu^7 + o(\mu^7), \qquad (5.228)$$

$$\psi_4(\mu) = 2(1+i)\alpha^2\mu^9 + i\alpha^2 G(a)\mu^8 - (1-i)\alpha^2\Big(\frac{1}{8}G^2(a) + \frac{3}{2}g(a)\Big)\mu^7 + o(\mu^7). \qquad (5.229)$$

On the other hand, it follows from Proposition 5.43 that

$$\tau_0 = -\frac{7\pi}{4a}. \qquad (5.230)$$

Substituting (5.228), (5.229), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\tau_1 = \frac{1}{4}\frac{G(a)}{\pi}, \tag{5.231}$$

$$\tau_2 = \frac{7}{16}\frac{G(a)}{\pi} + \frac{3a}{4}\frac{g(a)}{\pi^2}, \tag{5.232}$$

and the following result

**Theorem 5.44.** *For $g \in C^1[0,a]$, there exists a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, counted with multiplicity, of the problem (3.1)–(3.2), where $B_1(y) = y(0)$, $B_2(y) = y''(0)$, $B_3y = y'(a) - i\alpha\lambda y''(a)$ and $B_4y = y(a) + i\alpha\mu^2 y^{[3]}(a)$, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda}_k$ for $k \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}),$$

*where the numbers $\tau_0$, $\tau_1$ and $\tau_2$ are as respectively defined in (5.230), (5.231) and (5.232). In particular, there is an even number of imaginary eigenvalues.*

## 5.11 The boundary terms $B_1y$ and $B_2y$ are the following: $B_1y = y(0)$ and $B_2y = y'(0)$

In this section $\gamma_{1,k} = \delta_{k,0}(0,\mu)$ and $\gamma_{2,k} = \delta_{k,1}(0,\mu)$, where $k = 1, 2, 3, 4$. We give the functions $\psi_1$ and $\psi_4$ for each of different cases of $\gamma_{3,1}$, $\gamma_{3,2}$, $\gamma_{3,4}$, $\gamma_{4,1}$, $\gamma_{4,2}$ and $\gamma_{4,4}$, where $\gamma_{3,1}$, $\gamma_{3,2}$, $\gamma_{3,4}$, $\gamma_{4,1}$, $\gamma_{4,2}$ and $\gamma_{4,4}$ are as defined in (5.187), and we investigate the corresponding asymptotic eigenvalues.

It follows from (5.187) that

$$\gamma_{1,3}\gamma_{2,4} - \gamma_{2,3}\gamma_{1,4} = (1-i)\mu + \frac{1}{4}(1+i)g(0)\mu^{-1} + o(\mu^{-1}), \tag{5.233}$$

$$\gamma_{1,2}\gamma_{2,3} - \gamma_{2,2}\gamma_{1,3} = -(1+i)\mu - \frac{1}{4}(1-i)g(0)\mu^{-1} + o(\mu^{-1}). \tag{5.234}$$

### 5.11.1 Asymptotic of the eigenvalue for $B_3y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4y = y^{[3]}(a) - i\alpha\mu^2 y(a)$

Remark 5.36 and Proposition 4.36 lead to the following proposition

**Proposition 5.45.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y(0)$, $B_2(y) = y'(0)$, $B_3y = y''(a) + i\alpha\lambda y'(a)$ and $B_4y = y^{[3]}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\hat{\mu}_k^{\pm} = \pm(4k-1)\frac{\pi}{4a} + o(1), \ if \ k > 0$$
$$\hat{\mu}_k^{\pm} = \pm i(4|k|-1)\frac{\pi}{4a} + o(1), \ if \ k < 0$$

*as $|k| \to \infty$. In particular, there is an odd number of pure imaginary eigenvalues.*

In this subsection $\gamma_{3,k} = \delta_{k,2}(a,\mu) + i\alpha\mu^2\delta_{k,1}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,0}(a,\mu)$.

We calculate explicitly $\psi_1$ and $\psi_4$ as given respectively in (5.197) and (5.198). It follows from (5.187), (5.205), (5.206), (5.233) and (5.234)

$$\begin{aligned}\psi_1(\mu) &= 2(1-i)\alpha\mu^7 - (2(1+\alpha^2) + \alpha G(a))i\mu^6 \\ &\quad - 2(1+i)\Big(\frac{1}{4}(1+\alpha^2)G(a) + \frac{1}{16}\alpha G^2(a) - \frac{1}{4}\alpha g(0) + \alpha\Big)\mu^5 + o(\mu^5) \qquad (5.235)\\ \psi_4(\mu) &= 2(1+i)\alpha\mu^7 - (2(1+\alpha^2) - \alpha G(a))i\mu^6 \\ &\quad + 2(1-i)\Big(\frac{1}{4}(1+\alpha^2)G(a) - \frac{1}{16}\alpha G^2(a) + \frac{1}{4}\alpha g(0) - \alpha\Big)\mu^5 + o(\mu^5) \qquad (5.236)\end{aligned}$$

We recall that

$$\tau_0 = -\frac{\pi}{4a}, \tag{5.237}$$

see Proposition 5.45. Substituting (5.235), (5.236), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\tau_1 = \frac{i}{2}\frac{1+\alpha^2}{\alpha\pi} + \frac{1}{4}\frac{G(a)}{\pi} \tag{5.238}$$
$$\tau_2 = \frac{i}{8}\frac{1+\alpha^2}{\alpha\pi} - \frac{a((1-\alpha^2)^2 + \alpha^2 g(0))}{4\alpha^2\pi^2} + \frac{1}{16}\frac{G(a)}{\pi} \tag{5.239}$$

and the following result

**Theorem 5.46.** *Let $\tau_0$, $\tau_1$ and $\tau_2$ as respectively defined in* (5.237), (5.238) *and* (5.239). *Then for $g \in C^1[0,a]$ there is a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2), *where $B_1(y) = y(0)$, $B_2(y) = y'(0)$, $B_3y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4y = y^{[3]}(a) - i\alpha\mu^2 y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda_k}$ for $|k| \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}).$$

*In particular, there is an odd number of pure imaginary eigenvalues.*

### 5.11.2 Asymptotic of the eigenvalue for $B_3y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4y = y(a) + i\alpha\mu^2 y^{[3]}(a)$

Remark 5.36 and Proposition 4.38 lead to the following proposition

**Proposition 5.47.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2), *where $B_1(y) = y(0)$, $B_2(y) = y'(0)$, $B_3y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4y = y(a) + i\alpha\mu^2 y^{[3]}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\hat{\mu}_k^{\pm} = \pm(4k-5)\frac{\pi}{4a} + o(1), \quad if\ k > 0$$
$$\hat{\mu}_k^{\pm} = \pm i(4|k|-5)\frac{\pi}{4a} + o(1), \quad if\ k < 0.$$

*In particular, there is an even number of pure imaginary eigenvalues.*

In this section $\gamma_{3,k} = \delta_{k,2}(a,\mu) + i\alpha\mu^2\delta_{k,1}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,0}(a,\mu) + i\alpha\mu^2(\delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu))$. Putting (5.233), (5.234), (5.212) and (5.213) together, the functions $\psi_1$ and

$\psi_4$ given respectively in (5.197) and (5.198) are as follow

$$\begin{aligned}\psi_1(\mu) &= 2(1+i)\alpha^2\mu^9 + (\alpha^2 G(a) + 2\alpha)\mu^8 \\ &\quad + \frac{1}{2}(1-i)\alpha\Big(\frac{1}{4}\alpha G^2(a) - \alpha g(0) - G(a)\Big)\mu^7 + o(\mu^7), \qquad (5.240)\\ \psi_4(\mu) &= -2(1-i)\alpha^2\mu^9 - (\alpha^2 G(a) - 2\alpha)\mu^8 \\ &\quad - \frac{1}{2}(1+i)\alpha\Big(\frac{1}{4}\alpha G^2(a) - \alpha g(0) - G(a)\Big)\mu^7 + o(\mu^7). \qquad (5.241)\end{aligned}$$

It follows from Proposition 5.47 that

$$\tau_0 = -\frac{5\pi}{4a}. \qquad (5.242)$$

Substituting (5.240), (5.241), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\begin{aligned}\tau_1 &= \frac{i}{2\alpha\pi} + \frac{1}{4}\frac{G(a)}{\pi} \qquad (5.243)\\ \tau_2 &= \frac{5i}{8\alpha\pi} - \frac{a}{4\alpha^2\pi^2} - \frac{a}{\pi^2}\frac{g(0)}{4} + \frac{5}{16}\frac{G(a)}{\pi}, \qquad (5.244)\end{aligned}$$

and the following theorem

**Theorem 5.48.** *Let $\tau_0$, $\tau_1$ and $\tau_2$ as respectively defined in* (5.242), (5.243) *and* (5.244). *Then for $g \in C^1[0,a]$ there is a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2), *where $B_1(y) = y(0)$, $B_2(y) = y'(0)$, $B_3 y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4 y = y^{[3]}(a) - i\alpha\mu^2 y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda_k}$ for $|k| \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}).$$

*In particular, there is an even number of pure imaginary eigenvalues.*

### 5.11.3 Asymptotic of the eigenvalue for $B_3 y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4 y = y^{[3]}(a) - i\alpha\mu^2 y(a)$

Remark 5.36 and Proposition 4.41 lead to the following proposition

**Proposition 5.49.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y(0)$, $B_2(y) = y'(0)$, $B_3 y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4 y = y^{[3]}(a) - i\alpha\mu^2 y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\hat{\mu}_k^{\pm} = \pm(2k-1)\frac{\pi}{2a} + o(1), \quad if\ k > 0$$
$$\hat{\mu}_k^{\pm} = \pm i(2|k|-1)\frac{\pi}{2a} + o(1), \quad if\ k < 0.$$

*In particular, there is an odd number of pure imaginary eigenvalues.*

The terms $\gamma_{3,k}$ and $\gamma_{4,k}$ in this section are $\gamma_{3,k} = \delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,2}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,0}(a,\mu)$. It follows from (5.233), (5.234), (5.219) and (5.220) that the functions $\psi_1$ and $\psi_4$ given respectively in (5.197) and (5.198) are as follow

$$\begin{aligned}\psi_1(\mu) &= -2\alpha\mu^8 - (1-i)\Big(\frac{1}{2}\alpha G(a) + 2\alpha^2\Big)\mu^7 \\ &\quad + \Big(\frac{1}{8}G^2(a) + \alpha G(a) + \frac{3}{2}g(a) - \frac{1}{2}g(0)\Big)\alpha i\mu^6 + o(\mu^6), \qquad (5.245)\\ \psi_4(\mu) &= -2\alpha\mu^8 - (1+i)\Big(\frac{1}{2}\alpha G(a) - 2\alpha^2\Big)\mu^7 \\ &\quad - \Big(\frac{1}{8}G^2(a) - \alpha G(a) + \frac{3}{2}g(a) - \frac{1}{2}g(0)\Big)\alpha i\mu^6 + o(\mu^6). \qquad (5.246)\end{aligned}$$

On the other hand

$$\tau_0 = -\frac{\pi}{2a}, \qquad (5.247)$$

see Proposition 5.49.

Putting (5.245), (5.246), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\tau_1 = \frac{i\alpha}{\pi} + \frac{1}{4}\frac{G(a)}{\pi} \qquad (5.248)$$
$$\tau_2 = -\frac{a\alpha^2}{\pi^2} + \frac{i\alpha}{2\pi} - \frac{a}{4}\frac{g(0)}{\pi^2} + \frac{3a}{4}\frac{g(a)}{\pi^2} + \frac{1}{8\pi}G(a), \qquad (5.249)$$

and the following theorem

**Theorem 5.50.** *Let $\tau_0$, $\tau_1$ and $\tau_2$ as respectively defined in (5.247), (5.248) and (5.249). Then for $g \in C^1[0,a]$ there is a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y(0)$, $B_2(y) = y'(0)$, $B_3y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4y = y^{[3]}(a) - i\alpha\mu^2 y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda_k}$ for $|k| \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}).$$

*In particular, there is an odd number of pure imaginary eigenvalues.*

### 5.11.4 Asymptotic of the eigenvalue for $B_3y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4y = y(a) + i\alpha\mu^2 y^{[3]}(a)$

Remark 5.36 and Proposition 4.43 lead to the following proposition

**Proposition 5.51.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y(0)$, $B_2(y) = y'(0)$, $B_3y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4y = y(a) + i\alpha\mu^2 y^{[3]}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\hat{\mu}_k^{\pm} = \pm(2k-3)\frac{\pi}{2a} + o(1), \quad if\ k > 0$$
$$\hat{\mu}_k^{\pm} = \pm i(2|k|-3)\frac{\pi}{2a} + o(1), \quad if\ k < 0.$$

*In particular, there is an even number of pure imaginary eigenvalues.*

The terms $\gamma_{3,k}$ and $\gamma_{4,k}$ in this subsection are $\gamma_{3,k} = \delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,2}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,0}(a,\mu) + i\alpha\mu^2(\delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu))$. Putting (5.233), (5.234), (5.226) and (5.227) together, we get the functions $\psi_1$ and $\psi_4$ as respectively given in (5.197) and (5.198)

$$\psi_1(\mu) = -2i\alpha\mu^8 - \frac{1}{2}(1+i)\alpha^2 G(a)\mu^7 + \Big(\frac{1}{8}G^2(a) - \frac{3}{2}g(a) + \frac{1}{2}g(0)\Big)\alpha^2\mu^6 + o(\mu^6), \quad (5.250)$$
$$\psi_4(\mu) = -2i\alpha\mu^8 + \frac{1}{2}(1-i)\alpha^2 G(a)\mu^7 + \Big(\frac{1}{8}G^2(a) + \frac{3}{2}g(a) - \frac{1}{2}g(0)\Big)\alpha^2\mu^6 + o(\mu^6). \quad (5.251)$$

On the other hand, it follows from Proposition 5.51 that

$$\tau_0 = -\frac{3\pi}{2a}. \tag{5.252}$$

Substituting (5.250), (5.251), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\tau_1 = \frac{1}{4}\frac{G(a)}{\pi} \tag{5.253}$$

$$\tau_2 = -\frac{a}{4}\frac{g(0)}{\pi^2} + \frac{3a}{4}\frac{g(a)}{\pi^2} + \frac{3}{8}\frac{G(a)}{\pi}, \tag{5.254}$$

and the following theorem

**Theorem 5.52.** *Let $\tau_0$, $\tau_1$ and $\tau_2$ as respectively defined in (5.252), (5.253) and (5.254). Then for $g \in C^1[0,a]$ there is a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y(0)$, $B_2(y) = y'(0)$, $B_3y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4y = y(a) + i\alpha\mu^2 y^{[3]}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda_k}$ for $|k| \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}).$$

*In particular, there is an even number of pure imaginary eigenvalues.*

## 5.12 The boundary terms $B_1y$ and $B_2y$ are the following: $B_1y = y''(0)$ and $B_2y = y^{[3]}(0)$

In this section $\gamma_{1,k} = \delta_{k,2}(0,\mu)$ and $\gamma_{2,k} = \delta_{k,3}(0,\mu) - g(0)\delta_{k,1}(0,\mu)$, where $k = 1,2,3,4$. We give the functions $\psi_1$ and $\psi_4$ for each of different cases of $\gamma_{3,1}$, $\gamma_{3,2}$, $\gamma_{3,4}$, $\gamma_{4,1}$, $\gamma_{4,2}$ and $\gamma_{4,4}$, where $\gamma_{3,1}$, $\gamma_{3,2}$, $\gamma_{3,4}$, $\gamma_{4,1}$, $\gamma_{4,2}$ and $\gamma_{4,4}$ are as defined in (5.187), and we investigate the corresponding asymptotic eigenvalues.

It follows from (5.187) that

$$\gamma_{1,3}\gamma_{2,4} - \gamma_{2,3}\gamma_{1,4} = (1-i)\mu^5 + \frac{3}{4}(1+i)g(0)\mu^5 + o(\mu^5), \tag{5.255}$$

$$\gamma_{1,2}\gamma_{2,3} - \gamma_{2,2}\gamma_{1,3} = (1+i)\mu^5 - \frac{3}{4}(1-i)g(0)\mu^5 + o(\mu^5). \tag{5.256}$$

### 5.12.1 Asymptotic of the eigenvalue for $B_3y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4y = y^{[3]}(a) - i\alpha\mu^2 y(a)$

It follows from Remark 5.36 and Proposition 4.46 that

**Proposition 5.53.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat\lambda_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y''(0)$, $B_2(y) = y^{[3]}(0)$, $B_3y = y''(a) + i\alpha\lambda y'(a)$ and $B_4y = y^{[3]}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat\lambda_k$ are pure imaginary for $|k| < k_0$ and $\hat\lambda_{-k} = -\overline{\hat\lambda_k}$ for $|k| \geq k_0$, where $\hat\lambda_k = (\hat\mu_k^{\pm})^2$ with*

$$\hat\mu_k^{\pm} = \pm(4k-5)\frac{\pi}{4a} + o(1),\ \ if\ \ k > 0$$
$$\hat\mu_k^{\pm} = \pm i(4|k|-5)\frac{\pi}{4a} + o(1),\ \ if\ \ k < 0$$

*as $|k| \to \infty$. In particular, there is an odd number of pure imaginary eigenvalues.*

In this subsection, $\gamma_{3,k} = \delta_{k,2}(a,\mu) + i\alpha\mu^2\delta_{k,1}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,0}(a,\mu)$.

We calculate explicitly $\psi_1$ and $\psi_4$ as given respectively in (5.197) and (5.198). It follows from (5.187), (5.205), (5.206), (5.255) and (5.256) that

$$\begin{aligned}\psi_1(\mu) &= -2(1-i)\alpha\mu^{11} + (2(1+\alpha^2) + \alpha G(a))i\mu^{10}\\ &\quad + 2(1+i)\Big(\frac{1}{4}(1+\alpha^2)G(a) + \frac{1}{16}\alpha G^2(a) + \frac{3}{4}\alpha g(0) + \alpha\Big)\mu^9 + o(\mu^9), \qquad (5.257)\\ \psi_4(\mu) &= -2(1+i)\alpha\mu^{11} + (2(1+\alpha^2) - \alpha G(a))i\mu^{10}\\ &\quad - 2(1-i)\Big(\frac{1}{4}(1+\alpha^2)G(a) - \frac{1}{16}\alpha G^2(a) - \frac{3}{4}\alpha g(0) - \alpha\Big)\mu^9 + o(\mu^9). \qquad (5.258)\end{aligned}$$

It follows from Proposition 5.53 that

$$\tau_0 = -\frac{5\pi}{4a}. \qquad (5.259)$$

Substituting (5.257), (5.258), (5.200) and (5.201) in (5.202) and comparing the coefficients of

$k_1$ and $k_2$ we get

$$\tau_1 = \frac{i}{2}\frac{1+\alpha^2}{\alpha\pi} + \frac{1}{4}\frac{G(a)}{\pi} \tag{5.260}$$

$$\tau_2 = \frac{5i}{8}\frac{1+\alpha^2}{\alpha\pi} - \frac{a((1-\alpha^2)^2 - 3\alpha^2 g(0))}{4\alpha^2\pi^2} + \frac{5}{16}\frac{G(a)}{\pi} \tag{5.261}$$

and the following theorem

**Theorem 5.54.** *For $g \in C^1[0,a]$, there is a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2)*, where $B_1(y) = y''(0)$, $B_2(y) = y^{[3]}(0)$, $B_3y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4 y = y^{[3]}(a) - i\alpha\mu^2 y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda_k}$ for $|k| \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}),$$

*where the numbers $\tau_0$, $\tau_1$ and $\tau_2$ are as respectively defined in* (5.259), (5.260) *and* (5.261). *In particular there is an odd number of pure imaginary eigenvalues.*

### 5.12.2 Asymptotic of the eigenvalue for $B_3y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4 y = y(a) + i\alpha\mu^2 y^{[3]}(a)$

Remark 5.36 and Proposition 4.48 lead to

**Proposition 5.55.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2)*, where $B_1(y) = y''(0)$, $B_2(y) = y^{[3]}(0)$, $B_3y = y''(a) + i\alpha\lambda y'(a)$ and $B_4y = y(a) + i\alpha\lambda y^{[3]}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\hat{\mu}_k^{\pm} = \pm(4k-9)\frac{\pi}{4a} + o(1), \ if \ k > 0$$
$$\hat{\mu}_k^{\pm} = \pm i(4|k|-9)\frac{\pi}{4a} + o(1), \ if \ k < 0$$

*as $|k| \to \infty$. In particular, there is an even number of pure imaginary eigenvalues.*

In this subsection, $\gamma_{3,k} = \delta_{k,2}(a,\mu) + i\alpha\mu^2\delta_{k,1}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,0}(a,\mu) + i\alpha\mu^2(\delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu))$. Putting (5.255), (5.256), (5.212) and (5.213) into (5.197) and (5.198), we get

$$\begin{aligned}\psi_1(\mu) &= -(2+i)\alpha\mu^{13} - (\alpha^2G(a)+\alpha)\mu^{12}\\ &\quad - \frac{1}{2}(1-i)\alpha\Big(\frac{1}{4}\alpha G^2(a) + 3\alpha g(0) + G(a)\Big)\mu^{11} + o(\mu^{11}), \end{aligned} \tag{5.262}$$

$$\begin{aligned}\psi_4(\mu) &= (2-i)\alpha\mu^{13} + (\alpha^2G(a)-\alpha)\mu^{12}\\ &\quad + \frac{1}{2}(1+i)\alpha\Big(\frac{1}{4}\alpha G^2(a) + 3\alpha g(0) - G(a)\Big)\mu^{11} + o(\mu^{11}). \end{aligned} \tag{5.263}$$

It can be inferred from Proposition 5.55 that

$$\tau_0 = -\frac{9\pi}{4a}. \tag{5.264}$$

Substituting (5.262), (5.263), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\tau_1 = \frac{i}{2\alpha\pi} + \frac{1}{4}\frac{G(a)}{\pi} \tag{5.265}$$

$$\tau_2 = \frac{9i}{8\alpha\pi} - \frac{a}{4\alpha^2\pi^2} + \frac{3a}{4}\frac{g(0)}{\pi^2} + \frac{9}{16}\frac{G(a)}{\pi} \tag{5.266}$$

and the following theorem

**Theorem 5.56.** *For $g \in C^1[0,a]$, there is a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y''(0)$, $B_2(y) = y^{[3]}(0)$, $B_3y = y''(a) + i\alpha\mu^2y'(a)$ and $B_4y = y(a) + i\alpha\mu^2y^{[3]}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda}_k$ for $|k| \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}),$$

*where the numbers $\tau_0$, $\tau_1$ and $\tau_2$ are as respectively defined in* (5.264), (5.265) *and* (5.266). *In particular there is an even number of pure imaginary eigenvalues.*

### 5.12.3 Asymptotic of the eigenvalue for $B_3y = y'(a) - i\alpha\mu^2y''(a)$ and $B_4y = y^{[3]}(a) - i\alpha\mu^2y(a)$

From Remark 5.36 and Proposition 4.51 we have the following proposition

**Proposition 5.57.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y''(0)$, $B_2(y) = y^{[3]}(0)$, $B_3y = y'(a) - i\alpha\lambda y''(a)$ and $B_4y = y^{[3]}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\hat{\mu}_k^{\pm} = \pm(2k-3)\frac{\pi}{2a} + o(1), \ \ if \ \ k > 0$$
$$\hat{\mu}_k^{\pm} = \pm i(2|k|-3)\frac{\pi}{2a} + o(1), \ \ if \ \ k < 0$$

*as $|k| \to \infty$. In particular, there is an odd number of pure imaginary eigenvalues.*

The terms $\gamma_{3,k}$ and $\gamma_{4,k}$ of this subsection are $\gamma_{3,k} = \delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,2}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,0}(a,\mu)$.

It follows from (5.219), (5.220), (5.255) and (5.256) that the functions $\psi_1$ and $\psi_4$ as respectively given in (5.197) and (5.198) are as follow

$$\begin{aligned}\psi_1(\mu) &= 2\alpha\mu^{12} + (1-i)\Big(\frac{1}{2}\alpha G(a) + 2\alpha^2\Big)\mu^{11} \\ &\quad - \Big(\frac{1}{8}G^2(a) + \alpha G(a) + \frac{3}{2}g(0) + \frac{3}{2}g(a)\Big)i\alpha\mu^{10} + o(\mu^{10}), \qquad (5.267)\\ \psi_4(\mu) &= 2\alpha\mu^{12} + (1+i)\Big(\frac{1}{2}\alpha G(a) - 2\alpha^2\Big)\mu^{11} \\ &\quad + \Big(\frac{1}{8}G^2(a) - \alpha G(a) + \frac{3}{2}g(0) + \frac{3}{2}g(a)\Big)i\alpha\mu^{10} + o(\mu^{10}). \qquad (5.268)\end{aligned}$$

On the other hand Proposition 5.57 gives

$$\tau_0 = -\frac{3\pi}{2a}. \tag{5.269}$$

Putting (5.267), (5.268), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\tau_1 = \frac{i\alpha}{\pi} + \frac{1}{4}\frac{G(a)}{\pi} \tag{5.270}$$
$$\tau_2 = -\frac{a\alpha^2}{\pi^2} + \frac{3i\alpha}{2\pi} + \frac{3a}{4}\frac{g(0)}{\pi^2} + \frac{3a}{4}\frac{g(a)}{\pi^2} + \frac{3}{8}\frac{G(a)}{\pi}, \tag{5.271}$$

and the following theorem

**Theorem 5.58.** *For $g \in C^1[0,a]$, there is a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y''(0)$, $B_2(y) = y^{[3]}(0)$, $B_3y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4y = y^{[3]}(a) - i\alpha\mu^2 y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda}_k$ for $|k| \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}),$$

*where the numbers $\tau_0$, $\tau_1$ and $\tau_2$ are as respectively defined in (5.269), (5.270) and (5.271). In particular there is an odd number of pure imaginary eigenvalues.*

### 5.12.4 Asymptotic of the eigenvalue for $B_3y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4y = y(a) + i\alpha\mu^2 y^{[3]}(a)$

Remark 5.36 and Proposition 4.53 lead to the following proposition

**Proposition 5.59.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y''(0)$, $B_2(y) = y^{[3]}(0)$, $B_3y = y'(a) - i\alpha\lambda y''(a)$ and $B_4y = y(a) + i\alpha\lambda y^{[3]}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\hat{\mu}_k^{\pm} = \pm(2k-5)\frac{\pi}{2a} + o(1), \ \text{if } k > 0$$
$$\hat{\mu}_k^{\pm} = \pm i(2|k|-5)\frac{\pi}{2a} + o(1), \ \text{if } k < 0$$

*as $|k| \to \infty$. In particular, there is an even number of pure imaginary eigenvalues.*

The terms $\gamma_{3,k}$ and $\gamma_{4,k}$ in this subsection are as follow $\gamma_{3,k} = \delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,2}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,0}(a,\mu) + i\alpha\mu^2(\delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu))$.

Putting together (5.226), (5.227), (5.255) and (5.256), we get the functions $\psi_1$ and $\psi_4$ as respectively given in (5.197) and (5.198), which are

$$\psi_1(\mu) = 2i\alpha^2\mu^{12} + \frac{1}{2}(1+i)\alpha^2 G(a)\mu^{11} + \Big(\frac{1}{8}G^2(a) + \frac{3}{2}g(0) + \frac{3}{2}g(a)\Big)\alpha^2\mu^{10} + o(\mu^{10}), \quad (5.272)$$
$$\psi_4(\mu) = 2i\alpha^2\mu^{12} - \frac{1}{2}(1-i)\alpha^2 G(a)\mu^{11} - \Big(\frac{1}{8}G^2(a) + \frac{3}{2}g(0) + \frac{3}{2}g(a)\Big)\alpha^2\mu^{10} + o(\mu^{10}). \quad (5.273)$$

On the other hand, it follows from Proposition 5.59 that

$$\tau_0 = -\frac{5\pi}{2a}. \tag{5.274}$$

Substituting (5.272), (5.278), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\tau_1 = \frac{1}{4}\frac{G(a)}{\pi} \tag{5.275}$$

$$\tau_2 = \frac{3a}{4}\frac{g(0)}{\pi^2} + \frac{3a}{4}\frac{g(a)}{\pi^2} + \frac{5}{8}\frac{G(a)}{\pi}, \tag{5.276}$$

and the following theorem

**Theorem 5.60.** *For $g \in C^1[0,a]$, there is a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y''(0)$, $B_2(y) = y^{[3]}(0)$, $B_3y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4y = y(a) + i\alpha\mu^2 y^{[3]}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda}_k$ for $|k| \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}),$$

*where the numbers $\tau_0$, $\tau_1$ and $\tau_2$ are as respectively defined in* (5.274), (5.275) *and* (5.276). *In particular there is an even number of pure imaginary eigenvalues.*

## 5.13 The boundary terms $B_1y$ and $B_2y$ are the following: $B_1y = y'(0)$ and $B_2y = y^{[3]}(0)$

In this section $\gamma_{1,k} = \delta_{k,1}(0,\mu)$ and $\gamma_{2,k} = \delta_{k,3}(0,\mu) - g(0)\delta_{k,1}(0,\mu)$, where $k = 1,2,3,4$. We give the functions $\psi_1$ and $\psi_4$ for each of different cases of $\gamma_{3,1}$, $\gamma_{3,2}$, $\gamma_{3,4}$, $\gamma_{4,1}$, $\gamma_{4,2}$ and $\gamma_{4,4}$, where f $\gamma_{3,1}$, $\gamma_{3,2}$, $\gamma_{3,4}$, $\gamma_{4,1}$, $\gamma_{4,2}$ and $\gamma_{4,4}$ are as defined in (5.187), and we investigate the corresponding asymptotic eigenvalues.

It follows from (5.187) that

$$\gamma_{1,3}\gamma_{2,4} - \gamma_{2,3}\gamma_{1,4} = -2i\mu^4 + o(\mu^2) \tag{5.277}$$

$$\gamma_{1,2}\gamma_{2,3} - \gamma_{2,2}\gamma_{1,3} = -2i\mu^4 + o(\mu^2). \tag{5.278}$$

### 5.13.1 Asymptotic of the eigenvalue for $B_3y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4y = y^{[3]}(a) - i\alpha\mu^2 y(a)$

From Remark 5.36 and Proposition 4.56, it follows

**Proposition 5.61.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y'(0)$, $B_2(y) = y^{[3]}(0)$, $B_3y = y''(a) + i\alpha\lambda y'(a)$ and $B_4y = y^{[3]}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda_k}$ for $|k| \geq k_0$, where $\lambda_k = (\mu_k^{\pm})^2$ with*

$$\mu_k^{\pm} = \pm(k-1)\frac{\pi}{a} + o(1),\ \ if\ \ k > 0$$
$$\mu_k^{\pm} = \pm i(|k|-1)\frac{\pi}{4a} + o(1),\ \ if\ \ k < 0.$$

*In particular, there is an odd number of pure imaginary eigenvalues.*

In this subsection, $\gamma_{3,k} = \delta_{k,2}(a,\mu) + i\alpha\mu^2\delta_{k,1}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,0}(a,\mu)$.

We give explicitly $\psi_1$ and $\psi_4$ from (5.197) and (5.198). It follows from (5.187), (5.205), (5.206), (5.277) and (5.278) that

$$\begin{aligned}\psi_1(\mu) &= -4i\alpha\mu^{10} - (1+i)(2(1+\alpha^2) + \alpha G(a))\mu^9 \\ &\quad - ((1+\alpha^2)G(a) + \frac{1}{4}\alpha G^2(a) - 4\alpha)\mu^8 + o(\mu^8),\end{aligned} \tag{5.279}$$

$$\begin{aligned}\psi_4(\mu) &= 4i\alpha\mu^{10} + (1-i)(2(1+\alpha^2) - \alpha G(a))\mu^9 \\ &\quad + ((1+\alpha^2)G(a) - \frac{1}{4}\alpha G^2(a) - 4\alpha)\mu^8 + o(\mu^8).\end{aligned} \tag{5.280}$$

It follows from Proposition 5.61 that

$$\tau_0 = -\frac{\pi}{a}. \tag{5.281}$$

Substituting (5.279), (5.280), (5.200) and (5.201) in (5.202) and comparing the coefficients of

$k_1$ and $k_2$ we get

$$\tau_1 = \frac{i}{2}\frac{1+\alpha^2}{\alpha\pi} + \frac{1}{4}\frac{G(a)}{\pi} \tag{5.282}$$

$$\tau_2 = \frac{i}{2}\frac{1+\alpha^2}{\alpha\pi} - \frac{a(1-\alpha^2)^2}{4\alpha^2\pi^2} + \frac{1}{4}\frac{G(a)}{\pi} \tag{5.283}$$

and the following result

**Theorem 5.62.** *For $g \in C^1[0,a]$, there is a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y'(0)$, $B_2(y) = y^{[3]}(0)$, $B_3y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4y = y^{[3]}(a) - i\alpha\mu^2 y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda}_k$ for $|k| \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}),$$

*where the numbers $\tau_0$, $\tau_1$ and $\tau_2$ are as respectively defined in (5.281), (5.282) and (5.283). In particular there is an odd number of pure imaginary eigenvalues.*

### 5.13.2 Asymptotic of the eigenvalue for $B_3y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4y = y(a) + i\alpha\mu^2 y^{[3]}(a)$

Remark 5.36 and Proposition 4.58 lead to the following proposition

**Proposition 5.63.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y'(0)$, $B_2(y) = y^{[3]}(0)$, $B_3y = y''(a) + i\alpha\lambda y'(a)$ and $B_4y = y(a) + i\alpha\lambda y^{[3]}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\hat{\mu}_k^{\pm} = \pm(k-2)\frac{\pi}{a} + o(1), \;\; if \;\; k > 0$$
$$\hat{\mu}_k^{\pm} = \pm i(|k|-2)\frac{\pi}{a} + o(1), \;\; if \;\; k < 0$$

*as $|k| \to \infty$. In particular, there is an even number of pure imaginary eigenvalues.*

In this subsection, $\gamma_{3,k} = \delta_{k,2}(a,\mu) + i\alpha\mu^2\delta_{k,1}(a,\mu)$ and $\gamma_{4,k} = \delta_{0,k}(a,\mu) + i\alpha\mu^2(\delta_{3,k}(a,\mu) - g(a)\delta_{k,1}(a,\mu))$. It follows from (5.277), (5.278), (5.212) and (5.213) that the functions $\psi_1$ and $\psi_4$ as given in (5.197) and (5.198) are as follow

$$\psi_1(\mu) = 4\alpha^2\mu^{12} + (1-i)(\alpha^2 G(a) + 2\alpha)\mu^{11} - \Big(\frac{1}{4}\alpha^2 G^2(a) + \alpha G(a)\Big)i\mu^{10} + o(\mu^{10}), \quad (5.284)$$

$$\psi_4(\mu) = -4\alpha^2\mu^{12} - (1+i)(\alpha^2 G(a) - 2\alpha)\mu^{11} - \Big(\frac{1}{4}\alpha^2 G^2(a) - \alpha G(a)\Big)i\mu^{10} + o(\mu^{10}). \quad (5.285)$$

It can be seen from Proposition 5.63 that

$$\tau_0 = -\frac{2\pi}{a}. \quad (5.286)$$

Putting (5.284), (5.285), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\tau_1 = \frac{i}{2\alpha\pi} + \frac{1}{4}\frac{G(a)}{\pi} \quad (5.287)$$

$$\tau_2 = \frac{i}{\alpha\pi} - \frac{a}{4\alpha^2\pi^2} + \frac{1}{2}\frac{G(a)}{\pi} \quad (5.288)$$

and the following result

**Theorem 5.64.** *For $g \in C^1[0,a]$, there is a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the problem* (3.1)–(3.2), *where $B_1(y) = y'(0)$, $B_2(y) = y^{[3]}(0)$, $B_3 y = y''(a) + i\alpha\mu^2 y'(a)$ and $B_4 y = y(a) + i\alpha\mu^2 y^{[3]}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda}_k$ for $|k| \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}),$$

*where the numbers $\tau_0$, $\tau_1$ and $\tau_2$ are as respectively defined in* (5.286), (5.287) *and* (5.288). *In particular there is an even number of pure imaginary eigenvalues.*

### 5.13.3 Asymptotic of the eigenvalue for $B_3 y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4 y = y^{[3]}(a) - i\alpha\mu^2 y(a)$

From Remark 5.36 and Proposition 4.60, it follows

**Proposition 5.65.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y'(0)$, $B_2(y) = y^{[3]}(0)$, $B_3 y = y'(a) - i\alpha\lambda y''(a)$ and $B_4 y = y^{[3]}(a) - i\alpha\lambda y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\hat{\mu}_k^{\pm} = \pm(4k-5)\frac{\pi}{4a} + o(1), \ \text{if } k > 0$$
$$\hat{\mu}_k^{\pm} = \pm i(4|k|-5)\frac{\pi}{4a} + o(1), \ \text{if } k < 0$$

*as $|k| \to \infty$. In particular, there is an odd number of pure imaginary eigenvalues.*

In this subsection, $\gamma_{3,k} = \delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,2}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,0}(a,\mu)$.

It follows from (5.277), (5.278), (5.219) and (5.220) that the functions $\psi_1$ and $\psi_4$ as given in (5.197) and (5.198) are as follow

$$\begin{aligned}\psi_1(\mu) &= -2(1-i)\alpha\mu^{11} + (\alpha G(a) + 4\alpha^2)i\mu^{10} \\ &\quad + (1+i)\left(\frac{1}{8}G^2(a) + \alpha G(a) + \frac{3}{2}g(a)\right)\alpha\mu^9 + o(\mu^9),\end{aligned} \tag{5.289}$$

$$\begin{aligned}\psi_4(\mu) &= -2(1+i)\alpha\mu^{11} - (\alpha G(a) - 4\alpha^2)i\mu^{10} \\ &\quad + (1-i)\left(\frac{1}{8}G^2(a) - \alpha G(a) + \frac{3}{2}g(a)\right)\alpha\mu^9 + o(\mu^9).\end{aligned} \tag{5.290}$$

It can be inferred from Proposition 5.65

$$\tau_0 = -\frac{5\pi}{4a}. \tag{5.291}$$

Substituting (5.284), (5.285), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\tau_1 = \frac{i\alpha}{\pi} + \frac{1}{4}\frac{G(a)}{\pi} \tag{5.292}$$

$$\tau_2 = \frac{5i\alpha}{4\pi} - \frac{a\alpha^2}{\pi^2} + \frac{3a}{4}\frac{g(a)}{\pi^2} + \frac{5}{16}\frac{G(a)}{\pi}, \tag{5.293}$$

and the following theorem

**Theorem 5.66.** *For $g \in C^1[0,a]$, there is a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y'(0)$, $B_2(y) = y^{[3]}(0)$, $B_3 y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4 y = y^{[3]}(a) - i\alpha\mu^2 y(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$ and $\lambda_{-k} = -\overline{\lambda}_k$ for $|k| \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}),$$

*where the numbers $\tau_0$, $\tau_1$ and $\tau_2$ are as respectively defined in (5.291), (5.292) and (5.293). In particular there is an odd number of pure imaginary eigenvalues.*

### 5.13.4 Asymptotic of the eigenvalue for $B_3 y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4 y = y(a) + i\alpha\mu^2 y^{[3]}(a)$

We have Remark 5.36 and Proposition 4.62 the following proposition

**Proposition 5.67.** *For $g \in C^1[0,a]$ there exists a positive integer $k_0$ such that the eigenvalues $\hat{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y'(0)$, $B_2(y) = y^{[3]}(0)$, $B_3 y = y'(a) - i\alpha\lambda y''(a)$ and $B_4 y = y(a) + i\alpha\lambda y^{[3]}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k| < k_0$ and $\hat{\lambda}_{-k} = -\overline{\hat{\lambda}_k}$ for $|k| \geq k_0$, where $\hat{\lambda}_k = (\hat{\mu}_k^{\pm})^2$ with*

$$\hat{\mu}_k^{\pm} = \pm(4k-9)\frac{\pi}{4a} + o(1), \ if \ k > 0$$
$$\hat{\mu}_k^{\pm} = \pm i(4|k|-9)\frac{\pi}{4a} + o(1), \ if \ k < 0$$

*as $|k| \to \infty$. In particular, there is an even number of pure imaginary eigenvalues.*

In this subsection, $\gamma_{3,k} = \delta_{k,1}(a,\mu) - i\alpha\mu^2\delta_{k,2}(a,\mu)$ and $\gamma_{4,k} = \delta_{k,0}(a,\mu) + i\alpha\mu^2(\delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu))$.

Putting together (5.277), (5.278), (5.226) and (5.227), then the functions $\psi_1$ and $\psi_4$ as given in (5.197) and (5.198) are as follow

$$\psi_1(\mu) = -2(1+i)\alpha^2\mu^{11} - \alpha^2 G(a)\mu^{10} - (1-i)\Big(\frac{1}{8}G^2(a) + \frac{3}{2}g(a)\Big)\alpha^2\mu^9 + o(\mu^9), \quad (5.294)$$

$$\psi_4(\mu) = 2(1-i)\alpha^2\mu^{11} + \alpha^2 G(a)\mu^{10} + (1+i)\Big(\frac{1}{8}G^2(a) + \frac{3}{2}g(a)\Big)\alpha^2\mu^9 + o(\mu^9). \quad (5.295)$$

On the other hand, it follows from Proposition 5.67

$$\tau_0 = -\frac{9\pi}{4a}. \tag{5.296}$$

Substituting (5.294), (5.295), (5.200) and (5.201) in (5.202) and comparing the coefficients of $k_1$ and $k_2$ we get

$$\tau_1 = \frac{1}{4}\frac{G(a)}{\pi} \tag{5.297}$$

$$\tau_2 = \frac{3a}{4}\frac{g(a)}{\pi^2} + \frac{9}{16}\frac{G(a)}{\pi}, \tag{5.298}$$

and the following theorem

**Theorem 5.68.** *For $g \in C^1[0,a]$, there is a positive integer $k_0$ such that the eigenvalues $\lambda_k$, $k \in \mathbb{Z}$, of the problem (3.1)–(3.2), where $B_1(y) = y'(0)$, $B_2(y) = y^{[3]}(0)$, $B_3y = y'(a) - i\alpha\mu^2 y''(a)$ and $B_4y = y(a) + i\alpha\mu^2 y^{[3]}(a)$, counted with multiplicity, can be enumerated in such a way that the eigenvalues $\lambda_k$ are pure imaginary for $|k| < k_0$, $\lambda_{-k} = -\overline{\lambda_k}$ for $|k| \geq k_0$, where $\lambda_k = \mu_k^2$ and the $\mu_k$ have the asymptottics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2}),$$

*where the numbers $\tau_0$, $\tau_1$ and $\tau_2$ are as respectively defined in (5.296), (5.297) and (5.298). In particular there is an even number of pure imaginary eigenvalues.*

# Chapter 6

# Spectral asymptotics of a self-adjoint sixth order differential operator with one eigenvalue parameter dependent boundary condition

## 6.1 Introduction

We consider, on the interval $[0, a]$, the boundary eigenvalue problem defined by the differential equation

$$-y^{(6)} - (gy'')'' = \lambda^2 y, \tag{6.1}$$

and the boundary conditions

$$y(\lambda, 0) = 0, \tag{6.2}$$

$$y'(\lambda, 0) = 0, \tag{6.3}$$

$$y''(\lambda, 0) = 0, \tag{6.4}$$

$$y(\lambda, a) = 0, \tag{6.5}$$

$$y''(\lambda, a) = 0, \tag{6.6}$$

$$y^{(4)}(\lambda, a) - i\alpha\lambda y'(\lambda, a) = 0, \tag{6.7}$$

where $g \in C^2[0, a]$, $a > 0$ and $\alpha > 0$. We associate a quadratic operator pencil

$$L(\lambda, \alpha) = \lambda^2 M - i\alpha\lambda K - A \tag{6.8}$$

in the space $L_2(0, a) \oplus \mathbb{C}$ with this problem, where $K$ and $M$ are bounded self-adjoint operators with domains $\mathscr{D}(K) = \mathscr{D}(M) = L_2(0, a) \oplus \mathbb{C}$, and given by

$$K = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \text{ and } M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.9}$$

The operator $A$ acting in $L_2(0, a) \oplus \mathbb{C}$ with domain

$$\mathscr{D}(A) = \left\{ \widetilde{y} = \begin{pmatrix} y \\ -y'(a) \end{pmatrix} : y \in W_6^2(0, a), y(0) = y'(0) = y''(0) = y(a) = y''(a) = 0 \right\}, \tag{6.10}$$

$$\text{is given by} \quad A\widetilde{y} = \begin{pmatrix} -y^{(6)} - (gy'')'' \\ y^{(4)}(a) \end{pmatrix} \text{ for } \widetilde{y} \in \mathscr{D}(A). \tag{6.11}$$

We prove in the next section, by the means of definitions and properties introduced in Section 2.3, that $A$ is a self-adjoint operator. We investigate the spectral properties of the boundary eigenvalue problem (6.1)–(6.7) in Section 6.4, while we study the asymptotics of the eigenvalues for $g = 0$ in Section 6.5. Finally we present in Section 6.6 the results on the investigations conducted on the asymptotics of the eigenvalues for arbitrary $g$. We have used definitions and properties introduced in Chapter 2, Section 4.2, Section 4.3, Section 5.2 and Section 5.3 to conduct the work presented in this chapter.

## 6.2 The operator $A$ is self-adjoint

**Proposition 6.1.** *The operator $A$ is densely defined.*

*Proof.* Let $\widetilde{w} = \begin{pmatrix} w \\ c \end{pmatrix} \in L_2(0,a) \oplus \mathbb{C}$ such that $\langle \widetilde{y}, \widetilde{w} \rangle = 0$ for all $\widetilde{y} \in \mathscr{D}(A)$, where

$$\langle \widetilde{y}, \widetilde{w} \rangle = \int_0^a y(x)\overline{w}(x)dx - y'(a)\overline{c}.$$

Let $y \in C_0^\infty(0,a)$. Then $y'(a) = 0$ and $\widetilde{y} = \begin{pmatrix} y \\ 0 \end{pmatrix} \in \mathscr{D}(A)$, thus

$$\int_0^a y(x)\bar{w}(x)dx = 0 \quad \text{for all} \quad y \in C_0^\infty(0,a).$$

Hence $w = 0$.

Let the polynomial $y(x) = -4x^3 + \frac{7}{a}x^4 - \frac{3}{a^2}x^5$. Then $y(0) = y'(0) = y''(0) = y(a) = y''(a) = 0$ and $y'(a) = a^2 \neq 0$. Whence

$$\widetilde{y} = \begin{pmatrix} y \\ -y'(a) \end{pmatrix} \in \mathscr{D}(A).$$

Since $w = 0$, then $0 = \langle \widetilde{y}, \widetilde{w} \rangle = y'(a)\bar{c}$ thus $c = 0$ and it follows that $\widetilde{w} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence

$$\mathscr{D}(A)^\perp = \{0\}.$$

Therefore $A$ is densely defined. $\square$

**Proposition 6.2.** *$A$ is a symmetric operator.*

*Proof.* Let $\widetilde{y}, \widetilde{z} \in \mathscr{D}(A)$,

$$\begin{aligned} \langle A\widetilde{y}, \widetilde{z} \rangle &= -\int_0^a [y^{(6)}(x) + (gy'')''(x)]\bar{z}(x)dx - y^{(4)}(a)\bar{z}'(a) \\ &= -\int_0^a y^{(6)}(x)\bar{z}(x)dx - \int_0^a (gy''(x))''\bar{z}(x)dx - y^{(4)}(a)\bar{z}'(a). \end{aligned} \tag{6.12}$$

However

$$\int_0^a y^{(6)}(x)\bar{z}(x)dx = (y^{(6)}, z) \tag{6.13}$$

and

$$\int_0^a (gy''(x))''\bar{z}(x)dx = ((gy'')'', z). \tag{6.14}$$

Then,

$$\begin{aligned}
(y^{(6)}, z) &= [y^{(5)}(x)\bar{z}(x)]_0^a - \int_0^a y^{(5)}(x)\bar{z}'(x)dx \\
&= -[y^{(4)}(x)\bar{z}'(x)]_0^a + \int_0^a y^{(4)}(x)\bar{z}''(x)dx \quad (z(0) = z(a) = 0) \\
&= -[y^{(4)}(x)\bar{z}'(x)]_0^a + [y^{(3)}(x)\bar{z}''(x)]_0^a - \int_0^a y^{(3)}(x)\bar{z}^{(3)}(x)dx \\
&= -[y^{(4)}(x)\bar{z}'(x)]_0^a - \int_0^a y^{(3)}(x)\bar{z}^{(3)}(x)dx \quad (z''(a) = z''(0) = 0).
\end{aligned} \tag{6.15}$$

The sesquilinear form

$$\int_0^a y^{(3)}(x)\bar{z}^{(3)}(x)dx = -(y^{(6)}, z) - [y^{(4)}(x)\bar{z}'(x)]_0^a$$

is symmetric, thus

$$\int_0^a y^{(3)}(x)\bar{z}^{(3)}(x)dx = -(y, z^{(6)}) - [y'(x)\bar{z}^{(4)}(x)]_0^a. \tag{6.16}$$

Hence it follows from (6.15) and (6.16) that

$$(y^{(6)}, z) = (y, z^{(6)}) - [y^{(4)}(x)\bar{z}'(x)]_0^a + [y'(x)\bar{z}^{(4)}(x)]_0^a. \tag{6.17}$$

Since $y'(0) = z'(0) = 0$, then

$$(y^{(6)}, z) = (y, z^{(6)}) - y^{(4)}(a)\bar{z}'(a) + y'(a)\bar{z}^{(4)}(a). \tag{6.18}$$

Similarly since $z(0) = z(a) = y''(0) = y''(a) = 0$, we have

$$\begin{aligned}
((gy'')'', z) &= [(gy'')'(x)\bar{z}(x)]_0^a - \int_0^a (gy'')'(x)\bar{z}'(x)dx \\
&= -[(gy'')(x)\bar{z}'(x)]_0^a + \int_0^a y''(x)g\bar{z}''(x)dx \\
&= \int_0^a y''(x)g\bar{z}''(x)dx.
\end{aligned} \tag{6.19}$$

The sesquilinear form

$$\int_0^a y''(x)g\bar{z}''(x)dx = ((gy'')'', z)$$

is symmetric, so

$$\int_0^a y''(x)g\bar{z}''(x)dx = (y, (gz'')''). \tag{6.20}$$

Hence (6.19) and (6.20) yield

$$((gy'')'', z) = (y, (gz'')''). \tag{6.21}$$

Therefore it follows from (6.12), (6.13), (6.14), (6.18) and (6.21) that

$$\begin{aligned}
\langle A\widetilde{y}, \widetilde{z}\rangle &= -(y, z^{(6)}) + y^{(4)}(a)\bar{z}'(a) - y'(a)\bar{z}^{(4)}(a) - (y, (gz'')'') - y^{(4)}(a)\bar{z}'(a) \\
&= -(y, z^{(6)}) - (y, (gz'')'') - y'(a)\bar{z}^{(4)}(a) \\
&= (y, -z^{(6)} - (gz'')'') - y'(a)\bar{z}^{(4)}(a) \\
&= \int_0^a y(x)[-\bar{z}^{(6)}(x) - (g\bar{z}''(x))'']dx - y'(a)\bar{z}^{(4)}(a) \\
&= \langle \widetilde{y}, A\widetilde{z}\rangle.
\end{aligned} \tag{6.22}$$

Since $A$ is densely defined and $\langle A\widetilde{y}, \widetilde{z}\rangle = \langle \widetilde{y}, A\widetilde{z}\rangle$ for all $\widetilde{y}, \widetilde{z} \in \mathscr{D}(A)$, then according to Definition 2.18 $A$ is symmetric. $\square$

**Remark 6.3.** Let $g \in C^1[0,a]$ be a real valued function. Since the multiplication by $g$ is a continuous linear operator $g\cdot$ from $C^1[0,a]$ into itself, its adjoint $(g\cdot)^*$ from $(C_0^1(0,a))'$ into itself is well-defined. Let $(\cdot,\cdot)_{C_0^1(0,a)}$ be the sesquilinear form on $C_0^1(0,a) \times (C_0^1(0,a))'$. Note that for $f \in L_2(0,a)$ and $\phi \in C_0^1(0,a)$,

$$\begin{aligned}
(g\phi, f)_{C_0^1(0,a)} &= (g\phi, f) \\
&= (\phi, gf),
\end{aligned}$$

thus $(g\cdot)^* f = gf$. Hence we write

$$gu = (g\cdot)^* u \tag{6.23}$$

for all $u \in (C_0^1(0,a))'$.

Also note that we have the continuous embeddings with dense ranges

$$C_0^\infty(0,a) \hookrightarrow C_0^1(0,a) \hookrightarrow L_2(0,a), \tag{6.24}$$

whence

$$L_2(0,a) \hookrightarrow (C_0^1(0,a))' \hookrightarrow \mathscr{D}'(0,a). \tag{6.25}$$

In particular, $gu \in \mathscr{D}'(0,a)$ for all $u \in (C_0^1(0,a))'$.

**Lemma 6.4.** *If $z \in L_2(0,a)$ and $g \in C^1[0,a]$, then $gz' \in \mathscr{D}'(0,a)$.*

*Proof.* Since $z \in L_2(0,a) \hookrightarrow (C_0^1(0,a))' \hookrightarrow \mathscr{D}'(0,a)$ and $g \in C^1[0,a]$, then according to Remark 6.3

$$gz \in L_2(0,a) \hookrightarrow \mathscr{D}'(0,a) \tag{6.26}$$

and by Definition 2.34

$$(gz)' \in \mathscr{D}'(0,a). \tag{6.27}$$

Since $g \in C^1[0,a]$, then $g' \in C[0,a]$. And as $z \in L_2(0,a)$, then

$$g'z \in L_2(0,a) \hookrightarrow \mathscr{D}'(0,a). \tag{6.28}$$

Thus (6.26) and (6.27) give

$$(gz)' - g'z \in \mathscr{D}'(0,a). \tag{6.29}$$

Hence for all $\phi \in C_0^\infty(0,a)$ we have

$$\begin{aligned}
(\phi, (gz)' - g'z)_{C_0^\infty(0,a)} &= (\phi, (gz)')_{C_0^\infty(0,a)} - (\phi, g'z)_{C_0^\infty(0,a)} \\
&= -(\phi', gz)_{C_0^\infty(0,a)} - (g'\phi, z)_{C_0(0,a)} \\
&= -(g\phi', z)_{C_0^1(0,a)} - (g'\phi, z)_{C_0(0,a)} \\
&= -(g\phi' + g'\phi, z)_{C_0(0,a)} \\
&= -((g\phi)', z)_{C_0(0,a)} \\
&= (g\phi, z')_{C_0^1(0,a)} \\
&= (\phi, gz')_{C_0^\infty(0,a)}.
\end{aligned}$$

Since for all $\phi \in C_0^\infty(0,a)$

$$(\phi, (gz)' - g'z)_{C_0^\infty(0,a)} = (\phi, gz')_{C_0^\infty(0,a)}, \tag{6.30}$$

then $gz' = (gz)' - g'z \in \mathscr{D}'(0,a)$. $\square$

**Remark 6.5.** Let $g \in C^2[0,a]$ be a real valued function. Since the multiplication by $g$ is a continuous linear operator $g\cdot$ from $C^2[0,a]$ into itself, its adjoint $(g\cdot)^*$ from $(C_0^2(0,a))'$ into itself is well-defined. Let $(\cdot,\cdot)_{C_0^2(0,a)}$ be the sesquilinear form on $C_0^2(0,a) \times (C_0^2(0,a))'$. Note that for $f \in L_2(0,a)$ and $\phi \in C_0^2(0,a)$,

$$\begin{aligned}(g\phi, f)_{C_0^2(0,a)} &= (g\phi, f)\\ &= (\phi, gf),\end{aligned}$$

thus $(g\cdot)^* f = gf$. Hence we write

$$gu = (g\cdot)^* u \tag{6.31}$$

for all $u \in (C_0^2(0,a))'$. In particular, $gu \in \mathscr{D}'(0,a)$ for all $u \in (C_0^2(0,a))'$, see (6.25).

**Lemma 6.6.** *If $z \in L_2(0,a)$ and $g \in C^2[0,a]$, then $gz'' \in \mathscr{D}'(0,a)$.*

*Proof.* As $z \in L_2(0,a) \hookrightarrow (C_0^2(0,a))' \hookrightarrow \mathscr{D}'(0,a)$ and $g \in C^2[0,a]$, then according to Remark 6.5

$$gz \in L_2(0,a) \hookrightarrow \mathscr{D}'(0,a) \tag{6.32}$$

and by Definition 2.34

$$(gz)'' \in \mathscr{D}'(0,a). \tag{6.33}$$

Since $g \in C^2[0,a]$, then $g'' \in C[0,a]$. And as $z \in L_2(0,a)$, then

$$g''z \in L_2(0,a) \hookrightarrow \mathscr{D}'(0,a). \tag{6.34}$$

Thus (6.33), (6.34) and Lemma 6.4 give

$$(gz)'' - g''z - 2g'z' \in \mathscr{D}'(0,a). \tag{6.35}$$

Let $\xi = (gz)'' - g''z - 2g'z'$. Then for all $\phi \in C_0^\infty(0,a)$ we have

$$\begin{aligned}
(\phi, \xi)_{C_0^\infty(0,a)} &= (\phi, (gz)'')_{C_0^\infty(0,a)} - (\phi, g''z)_{C_0^\infty(0,a)} - (\phi, 2g'z')_{C_0^\infty(0,a)} \\
&= (-1)^2(\phi'', gz)_{C_0^\infty(0,a)} - (g''\phi, z)_{C_0(0,a)} - (2g'\phi, z')_{C_0^1(0,a)} \\
&= (g\phi'', z)_{C_0^2(0,a)} - (g''\phi, z)_{C_0(0,a)} + (2(g'\phi)', z)_{C_0(0,a)} \\
&= (g\phi'' - g''\phi + 2(g'\phi)', z)_{C_0(0,a)} \\
&= (g\phi'' + g''\phi + 2g'\phi', z)_{C_0(0,a)} \\
&= ((g\phi)'', z)_{C_0(0,a)} \\
&= (-1)^2(g\phi, z'')_{C_0^2(0,a)} \\
&= (\phi, gz'')_{C_0^\infty(0,a)}.
\end{aligned}$$

Since for all $\phi \in C_0^\infty(0,a)$,

$$(\phi, (gz)'' - g''z - 2g'z')_{C_0^\infty(0,a)} = (\phi, gz'')_{C_0^\infty(0,a)}, \tag{6.36}$$

then it follows from (6.35) that $gz'' = (gz)'' - g''z - 2g'z' \in \mathscr{D}'(0,a)$. $\square$

**Proposition 6.7.** *Let* $\widetilde{z} = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A^*)$. *Then* $z \in W_6^2(0,a)$ *and there exists* $c \in \mathbb{C}$ *such that* $A^*\widetilde{z} = \begin{pmatrix} -z^{(6)} - (gz'')'' \\ c \end{pmatrix}$.

*Proof.* Since $\widetilde{z} \in \mathscr{D}(A^*)$, then according to Definition 2.17, there exists $\widetilde{w} = \begin{pmatrix} w \\ c \end{pmatrix} \in L_2(0,a) \oplus \mathbb{C}$ such that $\langle A\widetilde{y}, \widetilde{z}\rangle = \langle \widetilde{y}, \widetilde{w}\rangle$, for all $\widetilde{y} \in \mathscr{D}(A)$. Let $\widetilde{y} = \begin{pmatrix} y \\ 0 \end{pmatrix} \in C_0^\infty(0,a) \oplus \{0\}$. Since $y \in C_0^\infty(0,a)$, then $y(0) = y'(0) = y''(0) = y(a) = y''(a) = 0$ and $\widetilde{y} \in \mathscr{D}(A)$. Also $y'(a) = y^{(4)}(a) = 0$. Hence

$$\begin{aligned}
\langle A\widetilde{y}, \widetilde{z}\rangle &= (-y^{(6)} - (gy'')'', z)_{C_0^\infty(0,a)} + y^{(4)}(a)\bar{d} \\
&= -(y^{(6)}, z)_{C_0^\infty(0,a)} - ((gy'')'', z)_{C_0(0,a)}.
\end{aligned} \tag{6.37}$$

Since $z \in L_2(0,a)$, then $z \in \mathscr{D}'(0,a)$ and

$$(y^{(6)}, z)_{C_0^\infty(0,a)} = (-1)^6(y, z^{(6)})_{C_0^\infty(0,a)} = (y, z^{(6)})_{C_0^\infty(0,a)}. \tag{6.38}$$

As $z \in L_2(0,a)$ and $g \in C^2[0,a]$, then according to Lemma 6.6, $gz'' \in \mathscr{D}'(0,a)$ and

$$\begin{aligned}((gy'')'', z)_{C_0(0,a)} &= (-1)^2 (gy'', z'')_{C_0^2(0,a)} = (gy'', z'')_{C_0^2(0,a)} \\ &= (y'', gz'')_{C_0^\infty(0,a)} = (-1)^2 (y, (gz'')'')_{C_0^\infty(0,a)} = (y, (gz'')'')_{C_0^\infty(0,a)}. \quad (6.39)\end{aligned}$$

It follows from (6.37), (6.38) and (6.39) that

$$\begin{aligned}\langle A\widetilde{y}, \widetilde{z}\rangle &= -(y, z^{(6)})_{C_0^\infty(0,a)} - (y, (gz'')'')_{C_0^\infty(0,a)} \\ &= (y, -z^{(6)} - (gz'')'')_{C_0^\infty(0,a)}. \quad (6.40)\end{aligned}$$

On the other hand

$$\begin{aligned}\langle A\widetilde{y}, \widetilde{z}\rangle = \langle \widetilde{y}, \widetilde{w}\rangle &= \int_0^a y(x)\overline{w}(x)dx = (y, w)_{C_0^\infty(0,a)} - y'(a)\bar{c} \\ &= (y, w)_{C_0^\infty(0,a)}. \quad (6.41)\end{aligned}$$

It follows from (6.37), (6.40) and (6.41) that

$$(-y^{(6)} - (gy'')'', z)_{C_0^\infty(0,a)} = (y, -z^{(6)} - (gz'')'')_{C_0^\infty(0,a)} \quad (6.42)$$

and

$$-z^{(6)} - (gz'')'' = w \in L_2(0,a). \quad (6.43)$$

Let $v = -z^{(5)} - (gz'')'$. Then $v' = -z^{(6)} - (gz'')'' = w \in L_2(0,a)$. Let $u(x) = \int_0^x w(t)dt$. Then $u \in L_2(0,a)$ and therefore

$$u \in W_1^2(0,a), \quad (6.44)$$

and $v' - u' = 0$. Then according to Theorem 2.35 $v - u = c_1$, where $c_1$ is a constant. Hence $v = u + c_1 \in W_1^2(0,a)$ and $-z^{(5)} - (gz'')' \in W_1^2(0,a)$. Since $g \in C^2[0,a]$, then it follows from Proposition 2.43 that $z \in W_6^2(0,a)$. □

**Theorem 6.8.** *The operator $A$ is self-adjoint.*

*Proof.* Let $\widetilde{y} \in \mathscr{D}(A)$ and $\widetilde{z} = \begin{pmatrix} z \\ d \end{pmatrix} \in \mathscr{D}(A^*)$. Then

$$\begin{aligned}\langle A\widetilde{y}, \widetilde{z}\rangle &= (-y^{(6)} - (gy'')'', z) + y^{(4)}(a)\bar{d} \\ &= -(y^{(6)}, z) - ((gy'')'', z) + y^{(4)}(a)\bar{d}. \quad (6.45)\end{aligned}$$

Since $z \in W_6^2(0,a)$ by Proposition 6.7, then we have

$$\begin{aligned}(y^{(6)}, z) &= [y^{(5)}(x)\bar{z}(x)]_0^a - \int_0^a y^{(5)}(x)\bar{z}'(x)dx \\ &= [y^{(5)}(x)\bar{z}(x)]_0^a - [y^{(4)}(x)\bar{z}'(x)]_0^a + \int_0^a y^{(4)}(x)\bar{z}''(x)dx \\ &= [y^{(5)}(x)\bar{z}(x)]_0^a - [y^{(4)}(x)\bar{z}'(x)]_0^a + [y^{(3)}(x)\bar{z}''(x)]_0^a - \int_0^a y^{(3)}(x)\bar{z}^{(3)}(x)dx. \qquad (6.46)\end{aligned}$$

For $y, z \in W_6^2(0,a)$, the sesquilinear form

$$\int_0^a y^{(3)}(x)\bar{z}^{(3)}(x)dx = -(y^{(6)}, z) + [y^{(5)}(x)\bar{z}(x)]_0^a - [y^{(4)}(x)\bar{z}'(x)]_0^a + [y^{(3)}(x)\bar{z}''(x)]_0^a$$

is symmetric, thus we have

$$\begin{aligned}\int_0^a y^{(3)}(x)\bar{z}^{(3)}(x)dx &= -(y, z^{(6)}) + [y(x)\bar{z}^{(5)}(x)]_0^a - [y'(x)\bar{z}^{(4)}(x)]_0^a + [y''(x)\bar{z}^{(3)}(x)]_0^a \\ &= -(y, z^{(6)}) - y'(a)\bar{z}^{(4)}(a), \qquad (6.47)\end{aligned}$$

as $y(0) = y(a) = y'(0) = y''(0) = y''(a) = 0$, and it follows from (6.46) and (6.47) that

$$\begin{aligned}(y^{(6)}, z) = (y, z^{(6)}) &+ [y^{(5)}(x)\bar{z}(x)]_0^a - [y^{(4)}(x)\bar{z}'(x)]_0^a + [y^{(3)}(x)\bar{z}''(x)]_0^a \\ &+ y'(a)\bar{z}^{(4)}(a). \qquad (6.48)\end{aligned}$$

On the other hand

$$\begin{aligned}((gy'')'', z) &= [(gy'')'(x)\bar{z}(x)]_0^a - \int_0^a (gy'')'(x)\bar{z}'(x)dx \\ &= [(gy'')'(x)\bar{z}(x)]_0^a - [(gy'')(x)\bar{z}'(x)]_0^a + \int_0^a (gy'')(x)\bar{z}''(x)dx \\ &= [(gy'')'(x)\bar{z}(x)]_0^a + \int_0^a (gy'')(x)\bar{z}''(x)dx \quad (y''(0) = y''(a) = 0). \qquad (6.49)\end{aligned}$$

As $y, z \in W_6^2(0,a)$, the sesquilinear form

$$\int_0^a gy''(x)\bar{z}''(x)dx = ((gy'')'', z) - [(gy'')'(x)\bar{z}(x)]_0^a$$

is symmetric, then

$$\begin{aligned}\int_0^a y''(x)g\bar{z}''(x)dx &= (y, (gz'')'') - [y(x)(g\bar{z}'')'(x)]_0^a \\ &= (y, (gz'')'') \quad (y(0) = y(a) = 0), \qquad (6.50)\end{aligned}$$

and

$$((gy'')'', z) = (y, (gz'')'') + [(gy'')'(x)\bar{z}(x)]_0^a. \tag{6.51}$$

Hence it follows from (6.45), (6.48) and (6.51) that

$$\begin{aligned}\langle A\widetilde{y}, \widetilde{z}\rangle &= (y, -z^{(6)} - ((gz'')'') - [y^{(5)}(x)\bar{z}(x)]_0^a + [y^{(4)}(x)\bar{z}'(x)]_0^a - [y^{(3)}(x)\bar{z}''(x)]_0^a \\ &\quad - y'(a)\bar{z}^{(4)}(a) - [(gy'')'(x)\bar{z}(x)]_0^a + y^{(4)}(a)\bar{d}. \end{aligned}\tag{6.52}$$

However

$$\langle A\widetilde{y}, \widetilde{z}\rangle = \langle \widetilde{y}, A^*\widetilde{z}\rangle = \langle \widetilde{y}, \widetilde{w}\rangle \quad \text{where} \quad \widetilde{w} = \begin{pmatrix} w \\ c \end{pmatrix},$$

and according to Proposition 6.7 $w = -z^{(6)} - (gz'')''$. Thus

$$\langle A\widetilde{y}, \widetilde{z}\rangle = \langle \widetilde{y}, A^*\widetilde{z}\rangle = (y, -z^{(6)} - (gz'')'') - y'(a)\bar{c}.$$

Then

$$\begin{aligned}-y'(a)\bar{c} &= -[y^{(5)}(x)\bar{z}(x)]_0^a + [y^{(4)}(x)\bar{z}'(x)]_0^a - [y^{(3)}(x)\bar{z}''(x)]_0^a - y'(a)\bar{z}^{(4)}(a) \\ &\quad - [(gy'')'(x)\bar{z}(x)]_0^a + y^{(4)}(a)\bar{d},\end{aligned}$$

and it follows that

$$\begin{aligned}0 &= -y'(a)(\bar{z}^{(4)}(a) - \bar{c}) - y^{(5)}(a)\bar{z}(a) + y^{(5)}(0)\bar{z}(0) - y^{(4)}(0)\bar{z}'(0) - y^{(3)}(a)\bar{z}''(a) \\ &\quad + y^{(3)}(0)\bar{z}''(0) - (gy'')'(a)\bar{z}(a) + (gy'')'(0)\bar{z}(0) + y^{(4)}(a)(\bar{d} + \bar{z}'(a)).\end{aligned}\tag{6.53}$$

There exists a polynomial $y_1$ such that $y_1(0) = y_1'(0) = y_1''(0) = y_1(a) = y_1''(a) = 0$, $y_1'(a) \neq 0$ and $y_1^{(3)}(0) = y_1^{(3)}(a) = y_1^{(4)}(0) = y_1^{(4)}(a) = y_1^{(5)}(0) = y_1^{(5)}(a) = 0$. Since $y_1''(0) = y_1^{(3)}(0) = y_1''(a) = y_1^{(3)}(a) = 0$, then $(gy_1'')'(0) = (gy_1'')'(a) = 0$. Thus $\begin{pmatrix} y_1 \\ -y_1'(a) \end{pmatrix} \in \mathscr{D}(A)$ and

$$y_1'(a)(\bar{z}^{(4)}(a) - \bar{c}) = 0. \tag{6.54}$$

Since $y_1'(a) \neq 0$, then $\bar{z}^{(4)}(a) = \bar{c}$, so

$$c = z^{(4)}(a), \tag{6.55}$$

and (6.53) reduces to

$$\begin{aligned}&- y^{(5)}(a)\bar{z}(a) + y^{(5)}(0)\bar{z}(0) - y^{(4)}(0)\bar{z}'(0) - y^{(3)}(a)\bar{z}''(a)\\&+ y^{(3)}(0)\bar{z}''(0) - (gy'')'(a)\bar{z}(a) + (gy'')'(0)\bar{z}(0) + y^{(4)}(a)(\bar{d} + \bar{z}'(a)) = 0.\end{aligned} \tag{6.56}$$

Also there exists a polynomial $y_2$ such that $y_2(0) = y_2'(0) = y_2''(0) = y_2(a) = y_2''(a) = 0$ and $y_2^{(4)}(a) \neq 0$, $y_2^{(3)}(0) = y_2^{(3)}(a) = y_2^{(4)}(0) = y_2^{(5)}(0) = y_2^{(5)}(a) = 0$. As $y_2''(0) = y_2^{(3)}(0) = y_2''(a) = y_2^{(3)}(a) = 0$, then $(gy_2'')'(0) = (gy_2'')'(a) = 0$. Thus $\begin{pmatrix} y_2 \\ -y_2'(a) \end{pmatrix} \in \mathscr{D}(A)$ and

$$y_2^{(4)}(a)(\bar{d} + \bar{z}'(a)) = 0. \tag{6.57}$$

since $y_2^{(4)}(a) \neq 0$ then

$$d = -z'(a). \tag{6.58}$$

Hence (6.56) reduces to

$$\begin{aligned}&- y^{(5)}(a)\bar{z}(a) + y^{(5)}(0)\bar{z}(0) - y^{(4)}(0)\bar{z}'(0) - y^{(3)}(a)\bar{z}''(a)\\&+ y^{(3)}(0)\bar{z}''(0) - (gy'')'(a)\bar{z}(a) + (gy'')'(0)\bar{z}(0) = 0.\end{aligned} \tag{6.59}$$

There exists a polynomial $y_3$ such that $y_3(0) = y_3'(0) = y_3''(0) = y_3(a) = y_3''(a) = 0$ and $y_3^{(5)}(0) \neq 0$, $y_3^{(3)}(0) = y_3^{(3)}(a) = y_3^{(4)}(0) = y_3^{(5)}(a) = 0$. Since $y_3''(0) = y_3^{(3)}(0) = y_3''(a) = y_3^{(3)}(a) = 0$, then $(gy_3'')'(0) = (gy_3'')'(a) = 0$. Hence $\begin{pmatrix} y_3 \\ -y_3'(a) \end{pmatrix} \in \mathscr{D}(A)$ and

$$y_3^{(5)}(0)\bar{z}(0) = 0. \tag{6.60}$$

As $y_3^{(5)}(0) \neq 0$, then

$$z(0) = 0, \tag{6.61}$$

and (6.59) reduces to

$$- y^{(5)}(a)\bar{z}(a) - y^{(4)}(0)\bar{z}'(0) - y^{(3)}(a)\bar{z}''(a) + y^{(3)}(0)\bar{z}''(0) - (gy'')'(a)\bar{z}(a) = 0. \tag{6.62}$$

There exists a polynomial $y_4$, such that $y_4(0) = y_4'(0) = y_4''(0) = y_4(a) = y_4''(a) = 0$, $y_4^{(5)}(a) \neq 0$ and $y_4^{(3)}(0) = y_4^{(3)}(a) = y_4^{(4)}(0) = 0$. As $y_4''(a) = y_4^{(3)}(a) = 0$, then $(gy_4'')'(a) = 0$ so $\begin{pmatrix} y_4 \\ -y'(a) \end{pmatrix} \in \mathscr{D}(A)$ and

$$y_4^{(5)}(a)\bar{z}(a) = 0. \tag{6.63}$$

Since $y_4^{(5)}(a) \neq 0$, then

$$z(a) = 0, \tag{6.64}$$

and (6.62) reduces to

$$-y^{(4)}(0)\bar{z}'(0) - y^{(3)}(a)\bar{z}''(a) + y^{(3)}(0)\bar{z}''(0). \tag{6.65}$$

There exists a polynomial $y_5$, such that $y_5(0) = y_5'(0) = y_5''(0) = y_5(a) = y_5''(a) = 0$ and $y_5^{(4)}(0) \neq 0$, $y_5^{(3)}(0) = y_5^{(3)}(a) = 0$, thus $\begin{pmatrix} y_5 \\ -y_5'(a) \end{pmatrix} \in \mathscr{D}(A)$. As $y_5^{(4)}(0) \neq 0$, then (6.65) gives

$$z'(0) = 0 \tag{6.66}$$

and reduces to

$$-y^{(3)}(a)\bar{z}''(a) + y^{(3)}(0)\bar{z}''(0). \tag{6.67}$$

There exists a polynomial $y_6$, such that $y_6(0) = y_6'(0) = y_6''(0) = y_6(a) = y_6''(a) = 0$, $y_6^{(3)}(0) = 0$ and $y_6^{(3)}(a) \neq 0$, hence $\begin{pmatrix} y_6 \\ -y_6'(a) \end{pmatrix} \in \mathscr{D}(A)$. Since $y_6^{(3)}(a) \neq 0$,then

$$z''(a) = 0 \tag{6.68}$$

Finally there exists a polynomial $y_7$, such that $y_7(0) = y_7'(0) = y_7''(0) = y_7(a) = y_7''(a) = 0$, and $y_7^{(3)}(0) \neq 0$. Thus $\begin{pmatrix} y_7 \\ -y_7'(a) \end{pmatrix} \in \mathscr{D}(A)$ and

$$z''(0) = 0 \tag{6.69}$$

Hence, from (6.55), (6.58), (6.61), (6.64), (6.66), (6.68) and (6.69) we have

$$z(0) = z'(0) = z''(0) = z(a) = z''(a) = 0, \quad d = -z'(a) \quad \text{and} \quad c = z^{(4)}(a).$$

Thus $\widetilde{z} \in \mathscr{D}(A)$, so $\mathscr{D}(A^*) \subset \mathscr{D}(A)$. Since $A$ is symmetric, see Proposition 6.2, then it follows from Definition 2.18 that $A$ is self-adjoint. □

## 6.3 Resolvent set of the operator pencil $L(\cdot, \alpha)$

**Proposition 6.9.** *$M + K = I$ and $M|_{\mathscr{D}(A(U))} > 0$.*

The proof of this proposition is similar to the proof of Proposition 3.3.

**Definition 6.10.** Define the $5 \times 12$ matrix $V$ of rank 5 such that for $y \in W_6^2(0,a)$,

$$VY_R = \Big(y(0), y'(0), y''(0), y(a), y''(a)\Big)^\top,$$

where $Y_R = \begin{pmatrix} Y(0) \\ Y(a) \end{pmatrix}$ with $Y = \Big(y^{[0]}, y^{[1]}, y^{[2]}, y^{[3]}, y^{[4]}, y^{[5]}\Big)^\top$, see [37, (1.6)].

**Definition 6.11.** We formally denote the collection of the six boundary conditions (6.2)–(6.7) by $U$ and define the following operators related to $U$:

$$U_0 y = y^{(4)}(a) \text{ and } U_1 y = -y'(a), \quad y \in W_6^2(0,a). \tag{6.70}$$

**Definition 6.12.** Define $Z_1 = \big\{y \in W_6^2(0,a) : VY_R = 0\big\}$. Let $Z_2$ be the complementary space of $Z_1$ in $W_6^2(0,a)$ and define

$$\begin{aligned} p_U : W_6^2(0,a) &\longrightarrow Z_1 \\ y &\mapsto y_U \end{aligned} \tag{6.71}$$

be the projection from $W_6^2(0,a)$ onto $Z_1$ along $Z_2$.

Define

$$s : W_6^2(0,a) \longrightarrow L_2(0,a) \oplus C \tag{6.72}$$

$$y \mapsto \widetilde{y} = \begin{pmatrix} y \\ -y'(a) \end{pmatrix}, \tag{6.73}$$

and

$$\begin{aligned} r : W_6^2(0,a) &\longrightarrow \mathbb{C}^{12} \\ y &\mapsto r(y) = Y_R. \end{aligned} \tag{6.74}$$

**Remark 6.13.** We can observe that $Z_1$ is a closed finite codimensional subspace of $W_6^2(0,a)$ and the operator $p_U$ is bounded.

**Proposition 6.14.** *The linear map $r$ is surjective and bounded.*

The proof of the above proposition is similar to the proof of Proposition 3.5.

**Remark 6.15.** It follows from Definition 6.10 that $\operatorname{rank} V = 5$, thus $\dim(N(V)) = 12-5 = 7$. Since $Z_2$ is the complementary of $Z_1$ in $W_6^2(0,a)$, see Definition 6.12, then it follows from Remark 6.13 that $\dim Z_2 = 12 - 7 = 5$.

**Remark 6.16.** The quasi-derivatives associated with (6.1) are given by

$$y^{[0]} = y,\ y^{[1]} = y',\ y^{[2]} = y'',\ y^{[3]} = y^{(3)},$$
$$y^{[4]} = y^{(4)} + gy'',\ y^{[5]} = y^{(5)} + (gy'')',\ y^{[6]} = y^{(6)} + (gy'')''.$$

**Definition 6.17.** Choose and fix a bijective linear map

$$D_2 : \mathbb{C}^5 \longrightarrow Z_2. \tag{6.75}$$

Finally let

$$D_1 : W_6^2(0,a) \longrightarrow \mathbb{C}^5 \tag{6.76}$$

be defined by $D_1 = D_2^{-1}(I - p_U)$.

Define the boundary eigenvalue operator function

$$T(\lambda) : W_6^2(0,a) \longrightarrow L_2(0,a) \oplus \mathbb{C} \oplus \mathbb{C}^5$$

by

$$T(\lambda)y = \begin{pmatrix} -y^{[6]} - \lambda^2 y \\ i\alpha\lambda U_1 y + U_0 y \\ VY_R \end{pmatrix},$$

where $\lambda \in \mathbb{C}$ and $y^{[6]} = y^{(6)} + (gy'')''$.

**Proposition 6.18.** *The linear map*

$$VrD_2 : \mathbb{C}^5 \to \mathbb{C}^5$$

*is invertible.*

The proof of the above proposition is similar to the proof of Proposition 3.8 while the proof of the following proposition is similar to the proof of Proposition 3.9.

**Proposition 6.19.** *Let $\lambda \in \mathbb{C}$. Then the operator*

$$\begin{pmatrix} -I & E_{12} \\ 0 & VrD_2 \end{pmatrix} : (L_2(0,a) \oplus \mathbb{C}) \oplus \mathbb{C}^5 \longrightarrow (L_2(0,a) \oplus \mathbb{C}) \oplus \mathbb{C}^5$$

*is bijective and bounded, where* $E_{12} = \begin{pmatrix} A_0 D_2 - \lambda^2 D_2 \\ U_0 D_2 + i\alpha\lambda U_1 D_2 \end{pmatrix}$ *and for* $y \in W_6^2(0,a)$, $A_0 y = -y^{[6]}$.

The proof of the following proposition is similar to the proof of Proposition 3.10.

**Proposition 6.20.** *Let $\lambda \in \mathbb{C}$ and $\mathscr{D}(L(\lambda,\alpha)) := \mathscr{D}(A(U))$. Then the operator*

$$\begin{pmatrix} sp_U \\ D_2^{-1}(I - p_U) \end{pmatrix} : W_4^2(0,a) \longrightarrow (L_2(0,a) \oplus \mathbb{C}) \oplus \mathbb{C}^5$$

*is injective and bounded with range $\mathscr{D}(L(\lambda,\alpha)) \oplus \mathbb{C}^5$.*

We refer the reader to the proof of Proposition 3.11 for the proof of the following proposition.

**Proposition 6.21.** *Let $\lambda \in \mathbb{C}$. Then* $T(\lambda) = \begin{pmatrix} -I & E_{12} \\ 0 & VrD_2 \end{pmatrix} \begin{pmatrix} L(\lambda,\alpha) & 0 \\ 0 & I_{\mathbb{C}^5} \end{pmatrix} \begin{pmatrix} sp_U \\ D_2^{-1}(I-p_U) \end{pmatrix}$.

**Corollary 6.22.** *The boundary eigenvalue operator function $T(\lambda)$ and the operator pencil $L(\cdot,\alpha)$ have the same resolvent set.*

The proof of the above corollary is similar to the proof of Corollary 3.12.

## 6.4 Spectrum of the differential operator

The proof of the following proposition is similar to the proof of Proposition 3.13.

**Proposition 6.23.** *The boundary eigenvalue operator function $T$ and therefore the operator pencil $L(\cdot,\alpha)$ is a Fredholm valued function with index 0. The spectrum of the Fredholm operator $T$ and therefore the Fredholm operator $L(\cdot,\alpha)$ consists of discrete eigenvalues of finite multiplicities and all eigenvalues of $L(\cdot,\alpha)$, $\alpha \geq 0$, lie in the closed upper half-plane and on the imaginary axis and are symmetric with respect to the imaginary axis.*

The proof of the following lemma is similar to the proof of Lemma 3.14.

**Lemma 6.24.** *All nonzero real eigenvalues of $L(\cdot,\alpha)$, $\alpha>0$, (if any) are semi-simple, i.e., the corresponding eigenvectors do not possess associated vectors. All real eigenvalues of $L(\cdot,\alpha)$, $\alpha>0$, are independent of $\alpha$.*

**Lemma 6.25.** *Let $\lambda=-i\tau$, $\tau>0$, be an eigenvalue of $L(\cdot,\alpha)$, $\alpha\geq 0$ . Then $\lambda$ is semi-simple.*

The proof of the above lemma is similar to the proof of Lemma 3.15.

**Proposition 6.26.** *Let $\lambda$ be an eigenvalue of $L(\cdot,\alpha)$, $\alpha\geq 0$. Then the geometric multiplicity of $\lambda$ is at most* 3.

*Proof.* Define

$$\begin{aligned} f_\lambda : N(L(\lambda,\alpha)) &\to \mathbb{C}^3 \\ y &\mapsto \begin{pmatrix} y^{(3)}(0) \\ y^{(4)}(0) \\ y^{(5)}(0) \end{pmatrix}. \end{aligned}$$

Let $y\in N(L(\lambda,\alpha))$ such that $f_\lambda(y)=0$. Then

$$\begin{cases} y^{(3)}(0)=0, \\ y^{(4)}(0)=0, \\ y^{(5)}(0)=0. \end{cases} \tag{6.77}$$

Since the three boundary conditions taken at the left endpoint 0 are independent of $\lambda$, then $y(0)=0$, $y'(0)=0$ and $y''(0)=0$, and it follows from (6.77) that $y=0$, thus $f_\lambda$ is injective. Hence the dimension of $N(L(\lambda,\alpha))$ is at most 3. Hence Proposition 6.26 follows. □

## 6.5 Asymptotics of eigenvalues for $g=0$

Let $B_1y=y(0)$, $B_2y=y'(0)$, $B_3y=y''(0)$, $B_4y=y(a)$, $B_5y=y''(a)$ and $B_6y=y^{(4)}(a)-i\alpha\lambda y'(a)$. Then it follows from the canonical fundamental system $y_j$, $j=$

$1, 2, 3, 4, 5, 6$ with $y_j^{(m)}(0) = \delta_{j,m+1}$ and $m = 0, 1, 2, 3, 4, 5$ that

$$\begin{cases} B_1 y_1 = y_1(0) = 1 \\ B_1 y_2 = y_2(0) = 0 \\ B_1 y_3 = y_3(0) = 0 \\ B_1 y_4 = y_4(0) = 0 \\ B_1 y_5 = y_5(0) = 0 \\ B_1 y_6 = y_6(0) = 0 \end{cases} \quad \begin{cases} B_2 y_1 = y_1'(0) = 0 \\ B_2 y_2 = y_2'(0) = 1 \\ B_2 y_3 = y_3'(0) = 0 \\ B_2 y_4 = y_4'(0) = 0 \\ B_2 y_5 = y_5'(0) = 0 \\ B_2 y_6 = y_6'(0) = 0 \end{cases} \quad \begin{cases} B_3 y_1 = y_1''(0) = 0 \\ B_3 y_2 = y_2''(0) = 0 \\ B_3 y_3 = y_3''(0) = 1 \\ B_3 y_4 = y_4''(0) = 0 \\ B_3 y_5 = y_5''(0) = 0 \\ B_3 y_6 = y_6''(0) = 0 \end{cases}.$$

that the characteristic matrix of this boundary problem is

$$M_c = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \end{pmatrix} \begin{pmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ B_4 y_1 & B_4 y_2 & B_4 y_3 & B_4 y_4 & B_4 y_5 & B_4 y_6 \\ B_5 y_1 & B_5 y_2 & B_5 y_3 & B_5 y_4 & B_5 y_5 & B_5 y_6 \\ B_6 y_1 & B_6 y_2 & B_6 y_3 & B_6 y_4 & B_6 y_5 & B_6 y_6 \end{pmatrix}.$$

The reduced characteristic matrix of the boundary value problem becomes

$$M = \begin{pmatrix} B_4 \\ B_5 \\ B_6 \end{pmatrix} \begin{pmatrix} y_4 & y_5 & y_6 \end{pmatrix} = \begin{pmatrix} B_4 y_4 & B_4 y_5 & B_4 y_6 \\ B_5 y_4 & B_5 y_5 & B_5 y_6 \\ B_6 y_4 & B_6 y_5 & B_6 y_6 \end{pmatrix}. \tag{6.78}$$

It is obvious that $\det M_c = \det M$. The characteristic function of the differential equation (6.1) for $g = 0$ is

$$\pi(\mu, \rho) = \rho^6 - \mu^6, \tag{6.79}$$

where $-\mu^6 = \lambda^2$, see (2.36). Thus $\lambda = \varepsilon i \mu^3$, where $\varepsilon \in \{-1, 1\}$. Let $\omega$, be the primitive sixth root of one. The zeros of the equation (6.79) are $\mu$, $\omega\mu$ $\omega^2\mu$, $\omega^3\mu$, $\omega^4\mu$ $\omega^5\mu$. Thus a fundamental system and a fundamental matrix of the equation (6.1), for $g = 0$, are respectively $\left\{e^{\mu x}, e^{\omega\mu x}, e^{\omega^2\mu x}, e^{\omega^3\mu x}, e^{\omega^4\mu x}, e^{\omega^5\mu x}\right\}$ and

$$Z(x,\mu)=\begin{pmatrix} e^{\mu x} & e^{\omega\mu x} & e^{\omega^2\mu x} & e^{\omega^3\mu x} & e^{\omega^4\mu x} & e^{\omega^5\mu x}\\ \mu e^{\mu x} & \mu\omega e^{\omega\mu x} & \mu\omega^2 e^{\omega^2\mu x} & \mu\omega^3 e^{\omega^3\mu x} & \mu\omega^4 e^{\omega^4\mu x} & \mu\omega^5 e^{\omega^5\mu x}\\ \mu^2 e^{\mu x} & \mu^2\omega^2 e^{\omega\mu x} & \mu^2\omega^4 e^{\omega^2\mu x} & \mu^2 e^{\omega^3\mu x} & \mu^2\omega^2 e^{\omega^4\mu x} & \mu^2\omega^4 e^{\omega^5\mu x}\\ \mu^3 e^{\mu x} & \mu^3\omega^3 e^{\omega\mu x} & \mu^3 e^{\omega^2\mu x} & \mu^3\omega^3 e^{\omega^3\mu x} & \mu^3 e^{\omega^4\mu x} & \mu^3\omega^3 e^{\omega^5\mu x}\\ \mu^4 e^{\mu x} & \mu^4\omega^4 e^{\omega\mu x} & \mu^4\omega^2 e^{\omega^2\mu x} & \mu^4 e^{\omega^3\mu x} & \mu^4\omega^4 e^{\omega^4\mu x} & \mu^4\omega^2 e^{\omega^5\mu x}\\ \mu^5 e^{\mu x} & \mu^5\omega^5 e^{\omega\mu x} & \mu^5\omega^4 e^{\omega^2\mu x} & \mu^5\omega^3 e^{\omega^3\mu x} & \mu^5\omega^2 e^{\omega^4\mu x} & \mu^5\omega e^{\omega^5\mu x} \end{pmatrix},$$

where

$$\begin{cases} \omega=\frac{1}{2}+i\frac{\sqrt{3}}{2},\\ \omega^2=-\frac{1}{2}+i\frac{\sqrt{3}}{2},\\ \omega^3=-1,\\ \omega^4=-\frac{1}{2}-i\frac{\sqrt{3}}{2},\\ \omega^5=\frac{1}{2}-i\frac{\sqrt{3}}{2}. \end{cases} \tag{6.80}$$

Let $\{y_1,y_2,y_3,y_4,y_5,y_6\}$ be the canonical fundamental system of the differential equation (6.1). Set $Y:=(y_j^{(i-1)})_{i,j=1}^6$. Then there is a matrix $c\in C^6$ such that $Y(x,\mu)=Z(x,\mu)c$, see Definition 2.100, and $I_6=Y(0,\mu)=Z(0,\mu)c$, see Theorem 2.91. Hence $c=Z(0,\mu)^{-1}$. However

$$Z(0,\mu)=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1\\ \mu & \mu\omega & \mu\omega^2 & \mu\omega^3 & \mu\omega^4 & \mu\omega^5\\ \mu^2 & \mu^2\omega^2 & \mu^2\omega^4 & \mu^2 & \mu^2\omega^2 & \mu^2\omega^4\\ \mu^3 & \mu^3\omega^3 & \mu^3 & \mu^3\omega^3 & \mu^3 & \mu^3\omega^3\\ \mu^4 & \mu^4\omega^4 & \mu^4\omega^2 & \mu^4 & \mu^4\omega^4 & \mu^4\omega^2\\ \mu^5 & \mu^5\omega^5 & \mu^5\omega^4 & \mu^5\omega^3 & \mu^5\omega^2 & \mu^5\omega \end{pmatrix},$$

thus

$$c=\begin{pmatrix} \frac{1}{6} & \frac{1}{6\mu} & \frac{1}{6\mu^2} & \frac{1}{6\mu^3} & \frac{1}{6\mu^4} & \frac{1}{6\mu^5}\\ \frac{1}{6} & \frac{1-\sqrt{3}i}{12\mu} & -\frac{1+\sqrt{3}i}{12\mu^2} & -\frac{1}{6\mu^3} & -\frac{1-\sqrt{3}i}{12\mu^4} & \frac{1+\sqrt{3}i}{12\mu^5}\\ \frac{1}{6} & -\frac{1+\sqrt{3}i}{12\mu} & -\frac{1-\sqrt{3}i}{12\mu^2} & \frac{1}{6\mu^3} & -\frac{1+\sqrt{3}i}{12\mu^4} & -\frac{1-\sqrt{3}i}{12\mu}\\ \frac{1}{6} & -\frac{1}{6\mu} & \frac{1}{6\mu^2} & -\frac{1}{6\mu^3} & \frac{1}{6\mu^4} & -\frac{1}{6\mu^5}\\ \frac{1}{6} & -\frac{1-\sqrt{3}i}{12\mu} & -\frac{1+\sqrt{3}i}{12\mu^2} & \frac{1}{6\mu^3} & -\frac{1-\sqrt{3}i}{12\mu^4} & -\frac{1+\sqrt{3}i}{12\mu^5}\\ \frac{1}{6} & \frac{1+\sqrt{3}i}{12\mu} & -\frac{1-\sqrt{3}i}{12\mu} & -\frac{1}{6\mu^3} & -\frac{1+\sqrt{3}i}{12\mu^4} & \frac{1-\sqrt{3}i}{12\mu} \end{pmatrix}.$$

Let

$$c = (c_{ij})_{i,j=1}^{6}.$$

Then the canonical fundamental system of the differential equation (6.1) is

$$y_j = c_{1j}e^{\mu x} + c_{2j}e^{\omega\mu x} + c_{3j}e^{\omega^2\mu x} + c_{4j}e^{\omega^3\mu x} + c_{5j}e^{\omega^4\mu x} + c_{6j}e^{\omega^5\mu x}, \tag{6.81}$$

where $j = 1, 2, 3, 4, 5, 6$. It follows that

$$\begin{aligned}
\det\ M &= B_4y_4B_5y_5B_6y_6 - B_4y_4B_5y_6B_6y_5 + B_5y_4B_6y_5B_4y_6 - B_5y_4B_4y_5B_6y_6 \\
&\quad + B_6y_4B_4y_5B_5y_6 - B_6y_4B_5y_5B_4y_6 \\
&= -\frac{1}{72\mu^6}\Big( - \varepsilon\alpha e^{(1+\sqrt{3}i)\mu a} + \varepsilon\alpha e^{-(1+\sqrt{3}i)\mu a} - 3e^{(1+\sqrt{3}i)\mu a} - 3e^{-(1+\sqrt{3}i)\mu a} \\
&\quad + \varepsilon\alpha e^{(-1+\sqrt{3}i)\mu a} - \varepsilon\alpha e^{-(-1+\sqrt{3}i)\mu a} - 3e^{(-1+\sqrt{3}i)\mu a} - 3e^{-(-1+\sqrt{3}i)\mu a} \\
&\quad + 8\varepsilon\alpha e^{\frac{1}{2}(1+\sqrt{3}i)\mu a} - 8\varepsilon\alpha e^{-\frac{1}{2}(1+\sqrt{3}i)\mu a} - 8\varepsilon\alpha e^{\frac{1}{2}(-1+\sqrt{3}i)\mu a} + 8\varepsilon\alpha e^{-\frac{1}{2}(-1+\sqrt{3}i)\mu a} \\
&\quad + \varepsilon\alpha e^{2\mu a} - \varepsilon\alpha e^{-2\mu a} - 3e^{2\mu a} - 3e^{-2\mu a} - 8\varepsilon\alpha e^{\mu a} + 8\varepsilon\alpha e^{-\mu a} + 18\Big) \\
&= -\frac{1}{72\mu^6}\Big((\varepsilon\alpha - 3)e^{2\mu a} - (\varepsilon\alpha + 3)e^{-2\mu a} - (\varepsilon\alpha + 3)e^{(1+\sqrt{3}i)\mu a} + (\varepsilon\alpha - 3)e^{-(1+\sqrt{3}i)\mu a} \\
&\quad + (\varepsilon\alpha - 3)e^{(-1+\sqrt{3}i)\mu a} - (\varepsilon\alpha + 3)e^{-(-1+\sqrt{3}i)\mu a} + 8\varepsilon\alpha e^{\frac{1}{2}(1+\sqrt{3}i)\mu a} - 8\varepsilon\alpha e^{-\frac{1}{2}(1+\sqrt{3}i)\mu a} \\
&\quad - 8\varepsilon\alpha e^{\frac{1}{2}(-1+\sqrt{3}i)\mu a} + 8\varepsilon\alpha e^{-\frac{1}{2}(-1+\sqrt{3}i)\mu a} - 8\varepsilon\alpha e^{\mu a} + 8\varepsilon\alpha e^{-\mu a} + 18\Big),
\end{aligned} \tag{6.82}$$

we recall that $\varepsilon \in \{-1, 1\}$. Hence the characteristic function of (6.1)–(6.7) for $g = 0$ is

$$D_0(\mu) = -\frac{1}{72\mu^6}\sum_{m=1}^{13} C_m(\varepsilon, \alpha)e^{\omega_m\mu a}, \tag{6.83}$$

where

$$\begin{cases}
\omega_1 = 2,\ \omega_2 = 1 + \sqrt{3}i,\ \omega_3 = -1 + \sqrt{3}i,\ \omega_4 = -2,\ \omega_5 = -1 - \sqrt{3}i, \\
\omega_6 = 1 - \sqrt{3}i,\ \omega_7 = 1,\ \omega_8 = \frac{1}{2}(1 + \sqrt{3}i),\ \omega_9 = -\frac{1}{2}(1 - \sqrt{3}i), \\
\omega_{10} = -1,\ \omega_{11} = -\frac{1}{2}(1 + \sqrt{3}i),\ \omega_{12} = \frac{1}{2}(1 - \sqrt{3}i),\ \omega_{13} = 0,
\end{cases} \tag{6.84}$$

while

$$\begin{cases} C_1(\varepsilon,\alpha) = C_3(\varepsilon,\alpha) = C_5(\varepsilon,\alpha) = \varepsilon\alpha - 3, \\ C_2(\varepsilon,\alpha) = C_4(\varepsilon,\alpha) = C_6(\varepsilon,\alpha) = \varepsilon\alpha + 3, \\ C_7(\varepsilon,\alpha) = C_9(\varepsilon,\alpha) = C_{11}(\varepsilon,\alpha) = -8\varepsilon\alpha, \\ C_8(\varepsilon,\alpha) = C_{10}(\varepsilon,\alpha) = C_{12}(\varepsilon,\alpha) = 8\varepsilon\alpha, \\ C_{13}(\varepsilon,\alpha) = 18. \end{cases} \tag{6.85}$$

Let

$$\begin{aligned} \widetilde{D}_0(\mu) := D_0(\mu)e^{-2\mu a} = -\frac{1}{72\mu^6}\Big(&\varepsilon\alpha - 3 - (\varepsilon\alpha+3)e^{(-1+\sqrt{3}i)\mu a} \\ &+ \sum_{m=3}^{13} C_m(\varepsilon,\alpha)e^{(\omega_m-\omega_1)\mu a}\Big), \end{aligned} \tag{6.86}$$

where

$$\begin{cases} \omega_3 - \omega_1 = -3 + \sqrt{3}i,\ \omega_4 - \omega_1 = -4,\ \omega_5 - \omega_1 = -3 - \sqrt{3}i, \\ \omega_6 - \omega_1 = -1 - \sqrt{3}i,\ \omega_7 - \omega_1 = -1,\ \omega_8 - \omega_1 = -\frac{1}{2}(3 - \sqrt{3}i), \\ \omega_9 - \omega_1 = -\frac{1}{2}(5 - \sqrt{3}i),\ \omega_{10} - \omega_1 = -3, \\ \omega_{11} - \omega_1 = -\frac{1}{2}(5 + \sqrt{3}i),\ \omega_{12} - \omega_1 = -\frac{1}{2}(3 + \sqrt{3}i),\ \omega_{13} - \omega_1 = -2. \end{cases} \tag{6.87}$$

Thus $\arg(\omega_3-\omega_1) = \arg(\omega_8-\omega_1) = \frac{5\pi}{6}$, $\arg(\omega_4-\omega_1) = \arg(\omega_7-\omega_1) = \arg(\omega_{10}-\omega_1) = \arg(\omega_{13}-\omega_1) = \pi$, $\arg(\omega_5-\omega_1) = \frac{7\pi}{6}$, $\arg(\omega_6-\omega_1) = \frac{4\pi}{3}$, $\frac{18\pi}{20} < \arg(\omega_9-\omega_1) < \frac{19\pi}{20}$, $\frac{21\pi}{20} < \arg(\omega_{11}-\omega_1) < \frac{22\pi}{20}$, $\arg(\omega_{12}-\omega_1) = \frac{7\pi}{6}$. Hence for $m = 3,\dots,13$, $\arg(\omega_m-\omega_1) \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$. It follows that for $m = 3\dots,13$ and for $\arg\ \mu \in [-\frac{\pi}{12}, \frac{\pi}{12}]$, $\arg(\omega_m-\omega_1)\mu a \in [\frac{7\pi}{12}, \frac{17\pi}{12}] \subset (\frac{\pi}{2}, \frac{3\pi}{2})$. Therefore for $m = 3,\dots,13$ and $\arg\ \mu \in [-\frac{\pi}{12}, \frac{\pi}{12}]$, $|e^{(\omega_m-\omega_1)\mu a}| = e^{-\cos\frac{\pi}{12}|\mu|a} \le e^{-\sin\frac{\pi}{12}|\mu|a}$ and the terms $C_m(\varepsilon,\alpha)e^{(\omega_m-\omega_1)\mu a}$ where $m = 3,\dots,13$ can be absorbed by $\widetilde{C}_1(\varepsilon,\alpha) = \varepsilon\alpha - 3 + o(\mu^{-s})$ for any nonnegative integer $s$. On the other hand, replacing $\omega_1$ by $\omega_2$ in (6.86) gives

$$\begin{cases} \omega_3 - \omega_2 = -2,\ \omega_4 - \omega_2 = -3 - \sqrt{3}i,\ \omega_5 - \omega_2 = -2 - 2\sqrt{3}i, \\ \omega_6 - \omega_2 = -2\sqrt{3}i,\ \omega_7 - \omega_2 = -\sqrt{3}i,\ \omega_8 - \omega_2 = -\frac{1}{2}(1 + \sqrt{3}i), \\ \omega_9 - \omega_2 = -\frac{1}{2}(3 + \sqrt{3}i),\ \omega_{10} - \omega_2 = -2 - \sqrt{3}i,\ \omega_{11} - \omega_2 = -\frac{1}{2}(3 + \sqrt{3}i), \\ \omega_{12} - \omega_2 = -\frac{1}{2}(1 + \sqrt{3}i),\ \omega_{13} - \omega_2 = -1 - \sqrt{3}i. \end{cases} \tag{6.88}$$

Thus $\arg(\omega_3-\omega_2)=\pi$, $\arg(\omega_4-\omega_2)=\arg(\omega_9-\omega_2)=\arg(\omega_{11}-\omega_2)=\frac{7\pi}{6}$, $\arg(\omega_5-\omega_2)=\arg(\omega_8-\omega_2)=\arg(\omega_{12}-\omega_2)=\arg(\omega_{13}-\omega_2)=\frac{4\pi}{3}$, $\arg(\omega_6-\omega_2)=\arg(\omega_7-\omega_2)=\frac{3\pi}{2}$ and $\frac{7\pi}{10}<\arg(\omega_{10}-\omega_2)<\frac{8\pi}{10}$. Whence for $m=3,\ldots,13$ $\arg(\omega_m-\omega_2)\in[\pi,\frac{3\pi}{2}]$ and for $\arg\ \mu\in[-\frac{\pi}{4},-\frac{\pi}{12}]$, $\arg(\omega_m-\omega_2)\mu a\in[\frac{3\pi}{4},\frac{17\pi}{12}]\subset(\frac{\pi}{2},\frac{3\pi}{2})$. Thence for $m=3,\ldots,13$ and $\arg\ \mu\in[-\frac{\pi}{4},-\frac{\pi}{12}]$, $|e^{(\omega_m-\omega_2)\mu a}|=e^{-\cos\frac{\pi}{4}|\mu|a}<e^{-\sin\frac{\pi}{12}|\mu|a}$ and the terms $C_m(\varepsilon,\alpha)e^{(\omega_m-\omega_2)\mu a}$ where $m=3,\ldots,13$ can be absorbed by $\widetilde{C}_2(\varepsilon,\alpha)=\varepsilon\alpha+3+o(\mu^{-s})$ for any nonnegative integer $s$.

Hence for $\arg\ \mu\in[-\frac{\pi}{4},\frac{\pi}{12}]$,

$$\widetilde{D}_0(\mu)=-\frac{1}{72\mu^6}\Big(\varepsilon\alpha-3-(\varepsilon\alpha+3)e^{(-1+\sqrt{3}i)\mu a}+o(\mu^{-s})\Big). \tag{6.89}$$

Let

$$\widetilde{D}_j^0(\mu)=\frac{(-1)^{j+1}}{\mu^6}\Big(C_{j+1}(\varepsilon,\alpha)-C_{j+2}(\varepsilon,\alpha)e^{(\omega_{j+2}-\omega_{j+1})\mu a}\Big), \tag{6.90}$$

$$\widetilde{D}_j(\mu)=\frac{(-1)^{j+1}}{72\mu^6}\Big(C_{j+1}(\varepsilon,\alpha)-C_{j+2}(\varepsilon,\alpha)e^{(\omega_{j+2}-\omega_{j+1})\mu a}+\sum_{m=3}^{13}C_m(\varepsilon,\alpha)e^{(\omega_m-\omega_{j+1})\mu a}\Big), \tag{6.91}$$

where $j=0,1,2,3,4,5$, while $C_{j+1}(\varepsilon,\alpha)$ and $C_{j+2}(\varepsilon,\alpha)$ are as given in (6.85). Then

$$\widetilde{D}_0^0(\mu)=-\frac{1}{72\mu^6}\Big(\varepsilon\alpha-3-(\varepsilon\alpha+3)e^{(-1+\sqrt{3}i)\mu a}\Big), \tag{6.92}$$

$\widetilde{D}_j=D_0e^{-\omega_{j+1}\mu a}$ and in the sectors $[-\frac{\pi}{4}+\frac{j\pi}{3},\frac{\pi}{12}+\frac{j\pi}{3}]$, $j=0,1,2,3,4,5$

$$\widetilde{D}_j(\mu)=\frac{(-1)^{j+1}}{72\mu^6}\Big(C_{j+1}(\varepsilon,\alpha)-C_{j+2}(\varepsilon,\alpha)e^{(\omega_{j+2}-\omega_{j+1})\mu a}+o(\mu^{-s})\Big). \tag{6.93}$$

Since the function exp is periodic and has $2i\pi$ as period, then it follows from (6.92) that the zeros of $\widetilde{D}_0^0(\mu)$ in the sector $[-\frac{\pi}{4},\frac{\pi}{12}]$ are simple and are

$$\mu_k=\begin{cases}-\frac{1+\sqrt{3}i}{4a}\Big(\ln\left|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\right|+2ki\pi\Big) & \text{if } \alpha>3,\\ -\frac{1+\sqrt{3}i}{4a}\Big(\ln\left|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\right|+(2k+1)i\pi\Big) & \text{if } \alpha<3,\end{cases} \tag{6.94}$$

where $k\in\mathbb{Z}$. The hexagon $H_1$ of centre 0 and vertices $\omega_1$, $\omega_2$, $\omega_3$, $\omega_4$, $\omega_5$, $\omega_6$ is convex, it contains the hexagon $H_2$ of centre 0 and vertices $\omega_7$, $\omega_8$, $\omega_9$, $\omega_{10}$, $\omega_{11}$, $\omega_{12}$. The hexagon $H_1$ is invariant by the rotation of centre 0 and angle $\frac{\pi}{3}$. Thus the zeros of $\widetilde{D}_j^0(\mu)$ in the sectors

$[-\frac{\pi}{4}+\frac{n\pi}{3},\frac{\pi}{12}+\frac{n\pi}{3}]$ are

$$\mu_{k,n}=\begin{cases} -\frac{1+\sqrt{3}i}{4a}\Big(\ln\Big|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big|+2ki\pi\Big)e^{\frac{n\pi i}{3}} \text{ if } \alpha>3,\\ -\frac{1+\sqrt{3}i}{4a}\Big(\ln\Big|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big|+(2k+1)i\pi\Big)e^{\frac{n\pi i}{3}} \text{ if } \alpha<3,\end{cases} \tag{6.95}$$

if $n=0,2,4$, and

$$\mu_{k,n}=\begin{cases} -\frac{1+\sqrt{3}i}{4a}\Big(\ln\Big|\frac{\varepsilon\alpha+3}{\varepsilon\alpha-3}\Big|+2ki\pi\Big)e^{\frac{n\pi i}{3}} \text{ if } \alpha>3,\\ -\frac{1+\sqrt{3}i}{4a}\Big(\ln\Big|\frac{\varepsilon\alpha+3}{\varepsilon\alpha-3}\Big|+(2k+1)i\pi\Big)e^{\frac{n\pi i}{3}} \text{ if } \alpha<3,\end{cases} \tag{6.96}$$

if $n=1,3,5$. Hence the zeros of $\widetilde{D}_j^0(\mu)$ in the sectors $[-\frac{\pi}{4}+\frac{n\pi}{3},\frac{\pi}{12}+\frac{n\pi}{3}]$, $n=0,1,2,3,4,5$, are

$$\mu_{k,n}=\begin{cases} (-1)^{n+1}\frac{1+\sqrt{3}i}{4a}\Big(\ln\Big|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big|+2ki\pi\Big)e^{\frac{n\pi i}{3}}) \text{ if } \alpha>3,\\ (-1)^{n+1}\frac{1+\sqrt{3}i}{4a}\Big(\ln\Big|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big|+(2k+1)i\pi\Big)e^{\frac{n\pi i}{3}} \text{ if } \alpha<3,\end{cases} \tag{6.97}$$

where $n=0,1,2,3,4,5$. All the zeros of $\widetilde{D}_j(\mu)$ in the sectors $[-\frac{\pi}{4}+\frac{n\pi}{3},\frac{\pi}{12}+\frac{n\pi}{3}]$, $n=0,1,2,3,4,5$, have the same asymptotics as the zeros of $\widetilde{D}_j^0(\mu)$. Hence the asymptotics of the zeros of $\widetilde{D}_j(\mu)$ and therefore the asymptotics of the zeros of $D_0(\mu)$ in the sectors $[-\frac{\pi}{4}+\frac{n\pi}{3},\frac{\pi}{12}+\frac{n\pi}{3}]$, $n=0,1,2,3,4,5$, are

$$\hat{\mu}_{k,n}=\begin{cases} (-1)^{n+1}\frac{1+\sqrt{3}i}{4a}\Big(\ln\Big|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big|+2ki\pi\Big)e^{\frac{n\pi i}{3}}+o(k^{-s}) \text{ if } \alpha>3,\\ (-1)^{n+1}\frac{1+\sqrt{3}i}{4a}\Big(\ln\Big|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big|+(2k+1)i\pi\Big)e^{\frac{n\pi i}{3}}+o(k^{-s}) \text{ if } \alpha<3,\end{cases} \tag{6.98}$$

where $n=0,1,2,3,4,5$, $s$ is any nonnegative integer and $\mu_{k,n}=O(k)$.

**Remark 6.27.** As $\lambda=i\varepsilon\mu^3$, then it follows from (6.95) and (6.96) that $\hat{\lambda}_{k,0}=i\varepsilon\hat{\mu}_{k,n}^3$ for $n=0,2,4$ and $\hat{\lambda}_{k,1}=i\varepsilon\hat{\mu}_{k,n}^3$ for $n=1,3,5$, where arg $\hat{\mu}_{k,n}\in[-\frac{\pi}{4}+\frac{n\pi}{3},\frac{\pi}{12}+\frac{n\pi}{3}]$.

Whence we have the following result

**Proposition 6.28.** *For $g=0$ there exists a positive number $k_0$ such that the eigenvalues $\hat{\lambda}$, with sufficiently large modulus, of the problem* (6.1)–(6.7) *are*

$$\hat{\lambda}_{k,0}=\begin{cases} \frac{1}{8a^3}\Big(2k\pi-i\ln\Big|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big|+o(k^{-s})\Big)^3\varepsilon \text{ if } \alpha>3,\\ \frac{1}{8a^3}\Big((2k+1)\pi-i\ln\Big|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big|+o(k^{-s})\Big)^3\varepsilon \text{ if } \alpha<3,\end{cases}$$

*and*

$$\hat{\lambda}_{k,1}=\begin{cases} -\frac{1}{8a^3}\Big(2k\pi-i\ln\Big|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big|+o(k^{-s})\Big)^3\varepsilon \text{ if } \alpha>3,\\ -\frac{1}{8a^3}\Big((2k+1)\pi-i\ln\Big|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big|+o(k^{-s})\Big)^3\varepsilon \text{ if } \alpha<3,\end{cases}$$

*where $s$ is any nonnegative integer, $|k|\ge k_0$, with $\varepsilon\in\{-1,1\}$.*

## 6.6 Birkhoff regularity

Recall that $\lambda^2 = -\mu^6$. Then $p_i(\cdot,\mu) = \sum_{i=0}^{5} \mu^j \pi_{5-i,j}$, $(i = 0,1,2,3,4,5\ j = 0,1,2,3,4,5)$, see (2.35). Hence it follows from (2.35) and (2.38) that $\pi_{1,1} = \pi_{2,2} = \pi_{3,3} = \pi_{4,4} = \pi_{5,5} = 0$, while $\pi_{6,6} = 1$. Thus the characteristic function of (6.1) is $\pi(\rho) = \rho^6 + 1$, see (2.36), where the variable in (2.35) is $\mu$. The zeros of $\pi(\rho) = \rho^6 + 1$ are $\widetilde{\omega}_0 = \frac{\sqrt{3}+i}{2}$, $\widetilde{\omega}_1 = i$, $\widetilde{\omega}_2 = -\frac{\sqrt{3}-i}{2}$, $\widetilde{\omega}_3 = -\frac{\sqrt{3}+i}{2}$, $\widetilde{\omega}_4 = -i$, $\widetilde{\omega}_5 = \frac{\sqrt{3}-i}{2}$. Since $n_0 = 0$, then it follows from Proposition 5.19 that $l = 6$. It follows from Theorem 5.20.A that $\nu_1 = 0$, $\nu_2 = 1$, $\nu_3 = 2$, $\nu_4 = 3$, $\nu_5 = 4$ and $\nu_6 = 5$. Hence we can choose

$$C(x,\mu) = \operatorname{diag}\Big(1,\mu,\mu^2,\mu^3,\mu^4,\mu^5\Big)C_1(x), \tag{6.99}$$

with

$$C_1(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{\sqrt{3}+i}{2} & i & -\frac{\sqrt{3}-i}{2} & -\frac{\sqrt{3}+i}{2} & -i & \frac{\sqrt{3}-i}{2} \\ \frac{1+\sqrt{3}i}{2} & -1 & \frac{1-\sqrt{3}i}{2} & \frac{1+\sqrt{3}i}{2} & -1 & \frac{1-\sqrt{3}i}{2} \\ i & -i & i & -i & i & -i \\ -\frac{1-\sqrt{3}i}{2} & 1 & -\frac{1+\sqrt{3}i}{2} & -\frac{1-\sqrt{3}i}{2} & 1 & -\frac{1+\sqrt{3}i}{2} \\ -\frac{\sqrt{3}-i}{2} & i & \frac{\sqrt{3}+i}{2} & \frac{\sqrt{3}-i}{2} & -i & -\frac{\sqrt{3}+i}{2} \end{pmatrix}, \tag{6.100}$$

where $C_1$ is the matrix defined in (5.45). Thus

$$C(x,\mu) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{(\sqrt{3}+i)\mu}{2} & i\mu & -\frac{(\sqrt{3}-i)\mu}{2} & -\frac{(\sqrt{3}+i)\mu}{2} & -i\mu & \frac{(\sqrt{3}-i)\mu}{2} \\ \frac{(1+\sqrt{3}i)\mu^2}{2} & -\mu^2 & \frac{(1-\sqrt{3}i)\mu^2}{2} & \frac{(1+\sqrt{3}i)\mu^2}{2} & -\mu^2 & \frac{(1-\sqrt{3}i)\mu^2}{2} \\ i\mu^3 & -i\mu^3 & i\mu^3 & -i\mu^3 & i\mu^3 & -i\mu^3 \\ -\frac{(1-\sqrt{3}i)\mu^4}{2} & \mu^4 & -\frac{(1+\sqrt{3}i)\mu^4}{2} & -\frac{(1-\sqrt{3}i)\mu^4}{2} & \mu^4 & -\frac{(1+\sqrt{3}i)\mu^4}{2} \\ -\frac{(\sqrt{3}-i)\mu^5}{2} & i\mu^5 & \frac{(\sqrt{3}+i)\mu^5}{2} & \frac{(\sqrt{3}-i)\mu^5}{2} & -i\mu^5 & -\frac{(\sqrt{3}+i)\mu^5}{2} \end{pmatrix}. \tag{6.101}$$

The boundary matrix functions defined in (5.47) are given by

$$W^{(0)}(\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} C(0,\mu)$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{(\sqrt{3}+i)\mu}{2} & i\mu & -\frac{(\sqrt{3}-i)\mu}{2} & -\frac{(\sqrt{3}+i)\mu}{2} & -i\mu & \frac{(\sqrt{3}-i)\mu}{2} \\ \frac{(1+\sqrt{3}i)\mu^2}{2} & -\mu^2 & \frac{(1-\sqrt{3}i)\mu^2}{2} & \frac{(1+\sqrt{3}i)\mu^2}{2} & -\mu^2 & \frac{(1-\sqrt{3}i)\mu^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.102}$$

and

$$W^{(1)}(\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \varepsilon\alpha\mu^3 & 0 & 0 & 1 & 0 \end{pmatrix} C(a,\mu)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{(1+\sqrt{3}i)\mu^2}{2} & -\mu^2 & \frac{(1-\sqrt{3}i)\mu^2}{2} & \frac{(1+\sqrt{3}i)\mu^2}{2} & -\mu^2 & \frac{(1-\sqrt{3}i)\mu^2}{2} \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix}, \tag{6.103}$$

where

$$\begin{cases} \gamma_1 = \frac{1}{2}((\sqrt{3}\varepsilon\alpha - 1) + (\sqrt{3} + \varepsilon\alpha)i)\mu^4, \\ \gamma_2 = (1 + \varepsilon\alpha i)\mu^4, \\ \gamma_3 = -\frac{1}{2}((\sqrt{3}\varepsilon\alpha + 1) + (\sqrt{3} - \varepsilon\alpha)i)\mu^4, \\ \gamma_4 = -\frac{1}{2}((\sqrt{3}\varepsilon\alpha + 1) - (\sqrt{3} - \varepsilon\alpha)i)\mu^4, \\ \gamma_5 = (1 - \varepsilon\alpha i)\mu^4, \\ \gamma_6 = \frac{1}{2}((\sqrt{3}\varepsilon\alpha - 1) - (\sqrt{3} + \varepsilon\alpha)i)\mu^4. \end{cases} \tag{6.104}$$

It is obvious that $\gamma_4 = \overline{\gamma_3}$, $\gamma_5 = \overline{\gamma_2}$, $\gamma_6 = \overline{\gamma_1}$. Choosing $C_2(\mu) = \mathrm{diag}(1, \mu, \mu^2, 1, \mu^2, \mu^4)$ if $\alpha \geq 0$, we obtain according to (5.39), $C_2(\mu)^{-1}W^{(j)} = W_0^{(j)} + O(\mu^{-1})$, where $j = 0, 1$,

$$W_0^{(0)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{\sqrt{3}+i}{2} & i & -\frac{\sqrt{3}-i}{2} & -\frac{\sqrt{3}+i}{2} & -i & \frac{\sqrt{3}-i}{2} \\ \frac{1+\sqrt{3}i}{2} & -1 & \frac{1-\sqrt{3}i}{2} & \frac{1+\sqrt{3}i}{2} & -1 & \frac{1-\sqrt{3}i}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.105}$$

and

$$W_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{1+\sqrt{3}i}{2} & -1 & \frac{1-\sqrt{3}i}{2} & \frac{1+\sqrt{3}i}{2} & -1 & \frac{1-\sqrt{3}i}{2} \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 \end{pmatrix}, \tag{6.106}$$

with

$$\begin{cases} \delta_1 = \frac{1}{2}((\sqrt{3}\varepsilon\alpha - 1) + (\sqrt{3} + \varepsilon\alpha)i), \\ \delta_2 = (1 + \varepsilon\alpha i), \\ \delta_3 = -\frac{1}{2}((\sqrt{3}\varepsilon\alpha + 1) + (\sqrt{3} - \varepsilon\alpha)i), \\ \delta_4 = -\frac{1}{2}((\sqrt{3}\varepsilon\alpha + 1) - (\sqrt{3} - \varepsilon\alpha)i), \\ \delta_5 = (1 - \varepsilon\alpha i), \\ \delta_6 = \frac{1}{2}((\sqrt{3}\varepsilon\alpha - 1) - (\sqrt{3} + \varepsilon\alpha)i). \end{cases} \tag{6.107}$$

It is easy to check that

$$|W_0^{(0)}| = 3, \tag{6.108}$$

while

$$|W_0^{(1)}| = 1 + \varepsilon^2\alpha^2. \tag{6.109}$$

Thus

$$|W_0^{(0)}| + |W_0^{(1)}| = 4 + \varepsilon^2\alpha^2 < \infty. \tag{6.110}$$

On the other hand

$$|C_2^{-1}(\mu)W^{(0)}(\mu) - W_0^{(0)}| = 0 \tag{6.111}$$

and

$$|C_2^{-1}(\mu)W^{(1)}(\mu) - W_0^{(1)}| = 0. \tag{6.112}$$

Hence

$$|C_2^{-1}(\mu)W^{(0)}(\mu) - W_0^{(0)}| + |C_2^{-1}(\mu)W^{(1)}(\mu) - W_0^{(1)}| = O(\mu^{-1}) \quad \text{as } \mu \to \infty. \tag{6.113}$$

Whence the assumptions (5.35) and (5.36) are fulfilled for $j = 0, 1$. According to Proposition 5.13 the matrices $\Delta$ of the eigenvalue problem (6.1)–(6.7) are the following four $6 \times 6$ diagonal matrices with three consecutive ones and three consecutive zeros in the diagonal in a cycle arrangement:

$$\begin{cases} \Delta_1 = \operatorname{diag}(1,1,1,0,0,0), \\ \Delta_2 = \operatorname{diag}(1,1,0,0,0,1), \\ \Delta_3 = \operatorname{diag}(1,0,0,0,1,1), \\ \Delta_4 = \operatorname{diag}(0,0,0,1,1,1), \\ \Delta_5 = \operatorname{diag}(0,0,1,1,1,0), \\ \Delta_6 = \operatorname{diag}(0,1,1,1,0,0). \end{cases} \tag{6.114}$$

**Remark 6.29.** Let

$$W_0^{(0)}\Delta_j + W_0^{(1)}(I - \Delta_j), \tag{6.115}$$

where $\Delta_j,\ j = 1,2,3,4,5,6$ are the matrices defined in (6.114). It is easy to check that $\Delta_4 = I - \Delta_1$, $\Delta_5 = I - \Delta_2$, while $\Delta_6 = I - \Delta_3$, it follows that after a permutation of columns, the matrices (6.107) are block diagonal matrices consisting of $3 \times 3$ blocks taken from three consecutive columns (in the sense of cyclic arrangement) of the first three rows of $W_0^{(0)}$ and the last three rows of $W_0^{(1)}$, respectively for the eigenvalue problem (6.1)–(6.7) These matrices are obviously invertible for $\alpha \geq 0$ and for all six choices of $\Delta$. Thus it follows from Proposition 5.13 and Definition 5.21 that,

**Proposition 6.30.** *The eigenvalue problem* (6.1)–(6.7) *is Birkhoff regular for* $\alpha \geq 0$.

## 6.7 The canonical fundamental system of $-y^{(6)} - (gy'')'' = \lambda^2 y$

It follows from (5.49)–(5.51) that $n_0 = 0$, thus $l = 6$ see Theorem 5.23. Hence the matrix $M_2(\cdot, \mu)$ defined in (5.68) is reduced to $I_6$ and since $n_0 - 1 < 0$, then $h_{n_0-1} = 0$. Let

$$\widetilde{Q} := \widetilde{Q}_{22}.$$

As $n = 6$, then it follows from (5.49)–(5.51) that

$$\begin{cases} k_4 = g, \\ k_3 = 2g', \\ k_2 = g'', \\ k_0 = k_1 = k_5 = 0. \end{cases} \tag{6.116}$$

Since $k_5 = 0$ and $h_{n_0-1} = 0$, as $n_0 - 1 < 0$, then $Q^{[0]'} = 0$, thus it follows from (5.71) that

$$Q^{[0]} = I_6. \tag{6.117}$$

According to Theorem 5.23 ii), we have

$$\begin{cases} \omega_1 = 1, \\ \omega_2 = \exp(\frac{\pi i}{3}) = \frac{1+\sqrt{3}i}{2}, \\ \omega_3 = \exp(\frac{2\pi i}{3}) = -\frac{1-\sqrt{3}i}{2}, \\ \omega_4 = \exp(\pi i) = -1, \\ \omega_5 = \exp(\frac{4\pi i}{3}) = -\frac{1+\sqrt{3}i}{2}, \\ \omega_6 = \exp(\frac{5\pi i}{3}) = \frac{1-\sqrt{3}i}{2}. \end{cases} \tag{6.118}$$

Set $\Omega := \Omega_6 = \operatorname{diag}(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6)$, see (5.63). Then $\Omega_6^0 = I_6$ and

$$\Omega = \operatorname{diag}\Big(1, \frac{1+\sqrt{3}i}{2}, -\frac{1-\sqrt{3}i}{2}, -1, -\frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}\Big). \tag{6.119}$$

Set $Q := Q_{22}$ and

$$Q^{[1]} = \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{14}^1 & a_{15}^1 & a_{16}^1 \\ a_{21}^1 & a_{22}^1 & a_{23}^1 & a_{24}^1 & a_{25}^1 & a_{26}^1 \\ a_{31}^1 & a_{32}^1 & a_{33}^1 & a_{34}^1 & a_{35}^1 & a_{36}^1 \\ a_{41}^1 & a_{42}^1 & a_{43}^1 & a_{44}^1 & a_{45}^1 & a_{46}^1 \\ a_{51}^1 & a_{52}^1 & a_{53}^1 & a_{54}^1 & a_{55}^1 & a_{56}^1 \\ a_{61}^1 & a_{62}^1 & a_{63}^1 & a_{64}^1 & a_{65}^1 & a_{66}^1 \end{pmatrix}.$$

It follows from (5.76) and (6.116) that $\Omega Q^{[1]} - Q^{[1]}\Omega = Q^{[0]\prime}$, thus $\Omega Q^{[1]} - Q^{[1]}\Omega = 0$, see (6.117). But

$$\Omega Q^{[1]} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1+\sqrt{3}i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1-\sqrt{3}i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1+\sqrt{3}i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-\sqrt{3}i}{2} \end{pmatrix} \begin{pmatrix} a^1_{11} & a^1_{12} & a^1_{13} & a^1_{14} & a^1_{15} & a^1_{16} \\ a^1_{21} & a^1_{22} & a^1_{23} & a^1_{24} & a^1_{25} & a^1_{26} \\ a^1_{31} & a^1_{32} & a^1_{33} & a^1_{34} & a^1_{35} & a^1_{36} \\ a^1_{41} & a^1_{42} & a^1_{43} & a^1_{44} & a^1_{45} & a^1_{46} \\ a^1_{51} & a^1_{52} & a^1_{53} & a^1_{54} & a^1_{55} & a^1_{56} \\ a^1_{61} & a^1_{62} & a^1_{63} & a^1_{64} & a^1_{65} & a^1_{66} \end{pmatrix}$$

$$= \begin{pmatrix} a^1_{11} & a^1_{12} & a^1_{13} & a^1_{14} & a^1_{15} & a^1_{16} \\ \frac{(1+\sqrt{3}i)a^1_{21}}{2} & \frac{(1+\sqrt{3}i)a^1_{22}}{2} & \frac{(1+\sqrt{3}i)a^1_{23}}{2} & \frac{(1+\sqrt{3}i)a^1_{24}}{2} & \frac{(1+\sqrt{3}i)a^1_{25}}{2} & \frac{(1+\sqrt{3}i)a^1_{26}}{2} \\ -\frac{(1-\sqrt{3}i)a^1_{31}}{2} & -\frac{(1-\sqrt{3}i)a^1_{32}}{2} & -\frac{(1-\sqrt{3}i)a^1_{33}}{2} & -\frac{(1-\sqrt{3}i)a^1_{34}}{2} & -\frac{(1-\sqrt{3}i)a^1_{35}}{2} & -\frac{(1-\sqrt{3}i)a^1_{36}}{2} \\ -a^1_{41} & -a^1_{42} & -a^1_{43} & -a^1_{44} & -a^1_{45} & -a^1_{46} \\ -\frac{(1+\sqrt{3}i)a^1_{51}}{2} & -\frac{(1+\sqrt{3}i)a^1_{52}}{2} & -\frac{(1+\sqrt{3}i)a^1_{53}}{2} & -\frac{(1+\sqrt{3}i)a^1_{54}}{2} & -\frac{(1+\sqrt{3}i)a^1_{55}}{2} & -\frac{(1+\sqrt{3}i)a^1_{56}}{2} \\ \frac{(1-\sqrt{3}i)a^1_{61}}{2} & \frac{(1-\sqrt{3}i)a^1_{62}}{2} & \frac{(1-\sqrt{3}i)a^1_{63}}{2} & \frac{(1-\sqrt{3}i)a^1_{64}}{2} & \frac{(1-\sqrt{3}i)a^1_{65}}{2} & \frac{(1-\sqrt{3}i)a^1_{66}}{2} \end{pmatrix}$$

while

$$Q^{[1]}\Omega = \begin{pmatrix} a^1_{11} & a^1_{12} & a^1_{13} & a^1_{14} & a^1_{15} & a^1_{16} \\ a^1_{21} & a^1_{22} & a^1_{23} & a^1_{24} & a^1_{25} & a^1_{26} \\ a^1_{31} & a^1_{32} & a^1_{33} & a^1_{34} & a^1_{35} & a^1_{36} \\ a^1_{41} & a^1_{42} & a^1_{43} & a^1_{44} & a^1_{45} & a^1_{46} \\ a^1_{51} & a^1_{52} & a^1_{53} & a^1_{54} & a^1_{55} & a^1_{56} \\ a^1_{61} & a^1_{62} & a^1_{63} & a^1_{64} & a^1_{65} & a^1_{66} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1+\sqrt{3}i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1-\sqrt{3}i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1+\sqrt{3}i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-\sqrt{3}i}{2} \end{pmatrix}$$

$$= \begin{pmatrix} a^1_{11} & \frac{(1+\sqrt{3}i)a^1_{12}}{2} & -\frac{(1-\sqrt{3}i)a^1_{13}}{2} & -a^1_{14} & -\frac{(1+\sqrt{3}i)a^1_{15}}{2} & \frac{(1-\sqrt{3}i)a^1_{16}}{2} \\ a^1_{21} & \frac{(1+\sqrt{3}i)a^1_{22}}{2} & -\frac{(1-\sqrt{3}i)a^1_{23}}{2} & -a^1_{24} & -\frac{(1+\sqrt{3}i)a^1_{25}}{2} & \frac{(1-\sqrt{3}i)a^1_{26}}{2} \\ a^1_{31} & \frac{(1+\sqrt{3}i)a^1_{32}}{2} & -\frac{(1-\sqrt{3}i)a^1_{33}}{2} & -a^1_{34} & -\frac{(1+\sqrt{3}i)a^1_{35}}{2} & \frac{(1-\sqrt{3}i)a^1_{36}}{2} \\ a^1_{41} & \frac{(1+\sqrt{3}i)a^1_{42}}{2} & -\frac{(1-\sqrt{3}i)a^1_{43}}{2} & -a^1_{44} & -\frac{(1+\sqrt{3}i)a^1_{45}}{2} & \frac{(1-\sqrt{3}i)a^1_{46}}{2} \\ a^1_{51} & \frac{(1+\sqrt{3}i)a^1_{52}}{2} & -\frac{(1-\sqrt{3}i)a^1_{53}}{2} & -a^1_{54} & -\frac{(1+\sqrt{3}i)a^1_{55}}{2} & \frac{(1-\sqrt{3}i)a^1_{56}}{2} \\ a^1_{61} & \frac{(1+\sqrt{3}i)a^1_{62}}{2} & -\frac{(1-\sqrt{3}i)a^1_{63}}{2} & -a^1_{64} & -\frac{(1+\sqrt{3}i)a^1_{65}}{2} & \frac{(1-\sqrt{3}i)a^1_{66}}{2} \end{pmatrix}.$$

Let $A_1 := \Omega Q^{[1]} - Q^{[1]}\Omega$. Thus

$$A_1 = \begin{pmatrix} 0 & \frac{(1-\sqrt{3}i)a_{12}^1}{2} & \frac{(3-\sqrt{3}i)a_{13}^1}{2} & 2a_{14}^1 & \frac{(3+\sqrt{3}i)a_{15}^1}{2} & \frac{(1+\sqrt{3}i)a_{16}^1}{2} \\ -\frac{(1-\sqrt{3}i)a_{21}^1}{2} & 0 & a_{23}^1 & \frac{(3+\sqrt{3}i)a_{24}^1}{2} & (1+\sqrt{3}i)a_{25}^1 & \sqrt{3}ia_{26}^1 \\ -\frac{(3-\sqrt{3}i)a_{31}^1}{2} & -a_{32}^1 & 0 & \frac{(1+\sqrt{3}i)a_{34}^1}{2} & \sqrt{3}ia_{35}^1 & -(1-\sqrt{3}i)a_{36}^1 \\ -2a_{14}^1 & -\frac{(3+\sqrt{3}i)a_{42}^1}{2} & -\frac{(1+\sqrt{3}i)a_{43}^1}{2} & 0 & -\frac{(1-\sqrt{3}i)a_{45}^1}{2} & -\frac{(3-\sqrt{3}i)a_{46}^1}{2} \\ -\frac{(3+\sqrt{3}i)a_{51}^1}{2} & -(1+\sqrt{3}i)a_{52}^1 & -\sqrt{3}ia_{53}^1 & \frac{(1-\sqrt{3}i)a_{54}^1}{2} & 0 & -a_{56}^1 \\ -\frac{(1+\sqrt{3}i)a_{61}^1}{2} & -\sqrt{3}ia_{62}^1 & (1-\sqrt{3}i)a_{63}^1 & \frac{(3-\sqrt{3}i)a_{64}^1}{2} & a_{65}^1 & 0 \end{pmatrix}. \tag{6.120}$$

Since $\Omega Q^{[1]} - Q^{[1]}\Omega = 0$, then it results from (6.118) that $a_{12}^1 = a_{13}^1 = a_{14}^1 = a_{15}^1 = a_{16}^1 = 0$, $a_{21}^1 = a_{23}^1 = a_{24}^1 = a_{25}^1 = a_{26}^1 = 0$, $a_{31}^1 = a_{32}^1 = a_{34}^1 = a_{35}^1 = a_{36}^1 = 0$, $a_{41}^1 = a_{42}^1 = a_{43}^1 = a_{45}^1 = a_{46}^1 = 0$, $a_{51}^1 = a_{52}^1 = a_{53}^1 = a_{54}^1 = a_{56}^1 = 0$ and $a_{61}^1 = a_{62}^1 = a_{63}^1 = a_{64}^1 = a_{65}^1 = 0$, hence

$$Q^{[1]} = \begin{pmatrix} a_{11}^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33}^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44}^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55}^1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66}^1 \end{pmatrix}. \tag{6.121}$$

Set

$$Q^{[2]} := \begin{pmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{14}^2 & a_{15}^2 & a_{16}^2 \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & a_{24}^2 & a_{25}^2 & a_{26}^2 \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & a_{34}^2 & a_{35}^2 & a_{36}^2 \\ a_{41}^2 & a_{42}^2 & a_{43}^2 & a_{44}^2 & a_{45}^2 & a_{46}^2 \\ a_{51}^2 & a_{52}^2 & a_{53}^2 & a_{54}^2 & a_{55}^2 & a_{56}^2 \\ a_{61}^2 & a_{62}^2 & a_{63}^2 & a_{64}^2 & a_{65}^2 & a_{66}^2 \end{pmatrix}.$$

It follows from (5.76) and (6.116) that $\Omega Q^{[2]} - Q^{[2]}\Omega = Q^{[1]'} + \frac{1}{6}k_4\Omega\varepsilon\varepsilon^\top\Omega^{-2}Q^{[0]}$, where $k_4 = g$. Let $A_2 := \Omega Q^{[2]} - Q^{[2]}\Omega$. As in (6.120), we have

$$A_2 = \begin{pmatrix} 0 & \frac{(1-\sqrt{3}i)a_{12}^2}{2} & \frac{(3-\sqrt{3}i)a_{13}^2}{2} & 2a_{14}^2 & \frac{(3+\sqrt{3}i)a_{15}^2}{2} & \frac{(1+\sqrt{3}i)a_{16}^2}{2} \\ -\frac{(1-\sqrt{3}i)a_{21}^2}{2} & 0 & a_{23}^2 & \frac{(3+\sqrt{3}i)a_{24}^2}{2} & (1+\sqrt{3}i)a_{25}^2 & \sqrt{3}ia_{26}^2 \\ -\frac{(3-\sqrt{3}i)a_{31}^2}{2} & -a_{32}^2 & 0 & \frac{(1+\sqrt{3}i)a_{34}^2}{2} & \sqrt{3}ia_{35}^2 & -(1-\sqrt{3}i)a_{36}^2 \\ -2a_{41}^2 & -\frac{(3+\sqrt{3}i)a_{42}^2}{2} & -\frac{(1+\sqrt{3}i)a_{43}^2}{2} & 0 & -\frac{(1-\sqrt{3}i)a_{45}^2}{2} & -\frac{(3-\sqrt{3}i)a_{46}^2}{2} \\ -\frac{(3+\sqrt{3}i)a_{51}^2}{2} & -(1+\sqrt{3}i)a_{52}^2 & -\sqrt{3}ia_{53}^2 & \frac{(1-\sqrt{3}i)a_{54}^2}{2} & 0 & -a_{56}^2 \\ -\frac{(1+\sqrt{3}i)a_{61}^2}{2} & -\sqrt{3}ia_{62}^2 & (1-\sqrt{3}i)a_{63}^2 & \frac{(3-\sqrt{3}i)a_{64}^2}{2} & a_{65}^2 & 0 \end{pmatrix}. \tag{6.122}$$

It can be easily checked that

$$\varepsilon\varepsilon^\top = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{6.123}$$

$$\Omega^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1-\sqrt{3}i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1+\sqrt{3}i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1-\sqrt{3}i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+\sqrt{3}i}{2} \end{pmatrix} \tag{6.124}$$

and

$$\Omega^{-2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1+\sqrt{3}i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1-\sqrt{3}i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1+\sqrt{3}i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1-\sqrt{3}i}{2} \end{pmatrix}. \tag{6.125}$$

It follows from (6.117), (6.119), (6.123) and (6.125) that

$$\Omega\varepsilon\varepsilon^\top\Omega^{-2}Q^{[0]} = \begin{pmatrix} 1 & -\frac{1+\sqrt{3}i}{2} & -\frac{1-\sqrt{3}i}{2} & 1 & -\frac{1+\sqrt{3}i}{2} & -\frac{1-\sqrt{3}i}{2} \\ \frac{1+\sqrt{3}i}{2} & \frac{1-\sqrt{3}i}{2} & -1 & \frac{1+\sqrt{3}i}{2} & \frac{1-\sqrt{3}i}{2} & -1 \\ -\frac{1-\sqrt{3}i}{2} & 1 & -\frac{1+\sqrt{3}i}{2} & -\frac{1-\sqrt{3}i}{2} & 1 & -\frac{1+\sqrt{3}i}{2} \\ -1 & \frac{1+\sqrt{3}i}{2} & \frac{1-\sqrt{3}i}{2} & -1 & \frac{1+\sqrt{3}i}{2} & \frac{1-\sqrt{3}i}{2} \\ -\frac{1+\sqrt{3}i}{2} & -\frac{1-\sqrt{3}i}{2} & 1 & -\frac{1+\sqrt{3}i}{2} & -\frac{1-\sqrt{3}i}{2} & 1 \\ \frac{1-\sqrt{3}i}{2} & -1 & \frac{1+\sqrt{3}i}{2} & \frac{1-\sqrt{3}i}{2} & -1 & \frac{1+\sqrt{3}i}{2} \end{pmatrix}. \tag{6.126}$$

Let $B_2 := Q^{[1]'} + \frac{k_4}{6}\Omega\varepsilon\varepsilon^\top\Omega^{-2}Q^{[0]}$. It follows from (6.121) and (6.126) that

$$B_2 = \begin{pmatrix} a_{11}^{1'} + \frac{g}{6} & -\frac{(1+\sqrt{3}i)g}{12} & -\frac{(1-\sqrt{3}i)g}{12} & \frac{g}{6} & -\frac{(1+\sqrt{3}i)g}{12} & -\frac{(1-\sqrt{3}i)g}{12} \\ \frac{(1+\sqrt{3}i)g}{12} & a_{22}^{1'} + \frac{(1-\sqrt{3}i)g}{12} & -\frac{g}{6} & \frac{(1+\sqrt{3}i)g}{12} & \frac{(1-\sqrt{3}i)g}{12} & -\frac{g}{6} \\ -\frac{(1-\sqrt{3}i)g}{12} & \frac{g}{6} & a_{33}^{1'} - \frac{(1+\sqrt{3}i)g}{12} & -\frac{(1-\sqrt{3}i)g}{12} & \frac{g}{6} & -\frac{(1+\sqrt{3}i)g}{12} \\ -\frac{g}{6} & \frac{(1+\sqrt{3}i)g}{12} & \frac{(1-\sqrt{3}i)g}{12} & a_{44}^{1'} - \frac{g}{6} & \frac{(1+\sqrt{3}i)g}{12} & \frac{(1-\sqrt{3}i)g}{12} \\ -\frac{(1+\sqrt{3}i)g}{12} & -\frac{(1-\sqrt{3}i)g}{12} & \frac{g}{6} & -\frac{(1+\sqrt{3}i)g}{12} & a_{55}^{1'} - \frac{(1-\sqrt{3}i)g}{12} & \frac{g}{6} \\ \frac{(1-\sqrt{3}i)g}{12} & -\frac{g}{6} & \frac{(1+\sqrt{3}i)g}{12} & \frac{(1-\sqrt{3}i)g}{12} & -\frac{g}{6} & a_{66}^{1'} + \frac{(1+\sqrt{3}i)g}{12} \end{pmatrix}. \tag{6.127}$$

Let

$$G(x) = \int_0^x g(t)dt. \tag{6.128}$$

It results from (6.122) and (6.127) that

$$\begin{cases} a_{11}^1 = -\frac{G}{6}, \\ a_{22}^1 = -\frac{(1-\sqrt{3}i)G}{12}, \end{cases} \qquad \begin{cases} a_{33}^1 = \frac{(1+\sqrt{3}i)G}{12}, \\ a_{44}^1 = \frac{G}{6}, \end{cases} \qquad \begin{cases} a_{55}^1 = \frac{(1-\sqrt{3}i)G}{12}, \\ a_{66}^1 = -\frac{(1+\sqrt{3}i)G}{12}, \end{cases} \tag{6.129}$$

and

$$\begin{cases} \frac{1-\sqrt{3}i}{2}a_{12}^2 = -\frac{(1+\sqrt{3}i)g}{12}, \\ \frac{3-\sqrt{3}i}{2}a_{13}^2 = -\frac{(1-\sqrt{3}i)g}{12}, \\ 2a_{14}^2 = \frac{g}{6}, \\ \frac{3+\sqrt{3}i}{2}a_{15}^2 = -\frac{(1+\sqrt{3}i)g}{12}, \\ \frac{1+\sqrt{3}i}{2}a_{16}^2 = -\frac{(1-\sqrt{3}i)g}{12}, \end{cases} \qquad \begin{cases} -\frac{1-\sqrt{3}i}{2}a_{21}^2 = \frac{(1+\sqrt{3}i)g}{12}, \\ a_{23}^2 = -\frac{g}{6}, \\ \frac{3+\sqrt{3}i}{2}a_{24}^2 = \frac{(1+\sqrt{3}i)g}{12}, \\ (1+\sqrt{3}i)a_{25}^2 = \frac{(1-\sqrt{3}i)g}{12}, \\ \sqrt{3}ia_{26}^2 = -\frac{g}{6}, \end{cases} \tag{6.130}$$

$$\begin{cases} -\frac{3-\sqrt{3}i}{2}a_{31}^2 = -\frac{(1-\sqrt{3}i)g}{12}, \\ -a_{32}^1 = \frac{g}{6}, \\ \frac{1+\sqrt{3}i}{2}a_{34}^2 = -\frac{(1-\sqrt{3}i)g}{12}, \\ \sqrt{3}ia_{35}^2 = \frac{g}{6}, \\ -(1-\sqrt{3}i)a_{36}^2 = -\frac{(1+\sqrt{3}i)g}{12}, \end{cases} \qquad \begin{cases} -2a_{41}^2 = -\frac{g}{6}, \\ -\frac{3+\sqrt{3}i}{2}a_{42}^2 = \frac{(1+\sqrt{3}i)g}{12}, \\ -\frac{1+\sqrt{3}i}{2}a_{43}^2 = \frac{(1-\sqrt{3}i)g}{12}, \\ -\frac{1-\sqrt{3}i}{2}a_{45}^2 = \frac{(1+\sqrt{3}i)g}{12}, \\ -\frac{3-\sqrt{3}i}{2}a_{46}^2 = \frac{(1-\sqrt{3}i)g}{12}, \end{cases} \tag{6.131}$$

$$\begin{cases} -\frac{3+\sqrt{3}i}{2}a_{51}^2 = -\frac{(1+\sqrt{3}i)g}{12}, \\ -(1+\sqrt{3}i)a_{52}^2 = -\frac{(1-\sqrt{3}i)g}{12}, \\ -\sqrt{3}ia_{53}^2 = \frac{g}{6}, \\ \frac{1-\sqrt{3}i}{2}a_{54}^2 = -\frac{(1+\sqrt{3}i)g}{12}, \\ -a_{56}^2 = \frac{g}{6}, \end{cases} \qquad \begin{cases} -\frac{1+\sqrt{3}i}{2}a_{61}^2 = \frac{(1-\sqrt{3}i)g}{12}, \\ -\sqrt{3}ia_{62}^2 = -\frac{g}{6}, \\ (1-\sqrt{3}i)a_{63}^2 = \frac{(1+\sqrt{3}i)g}{12}, \\ \frac{3-\sqrt{3}i}{2}a_{64}^2 = \frac{(1-\sqrt{3}i)g}{12}, \\ a_{65}^2 = -\frac{g}{6}. \end{cases} \tag{6.132}$$

Thus

$$\begin{cases} a_{12}^2 = \frac{(1-\sqrt{3}i)g}{12}, \\ a_{13}^2 = -\frac{(3-\sqrt{3}i)g}{36}, \\ a_{14}^2 = \frac{g}{12}, \\ a_{15}^2 = -\frac{(3+\sqrt{3}i)g}{36}, \\ a_{16}^2 = \frac{(1+\sqrt{3}i)g}{12}, \end{cases} \qquad \begin{cases} a_{21}^2 = \frac{(1-\sqrt{3}i)g}{12}, \\ a_{23}^2 = -\frac{g}{6}, \\ a_{24}^2 = \frac{(3+\sqrt{3}i)g}{36}, \\ a_{25}^2 = -\frac{(1+\sqrt{3}i)g}{24}, \\ a_{26}^2 = \frac{\sqrt{3}ig}{18}, \end{cases} \qquad \begin{cases} a_{31}^2 = \frac{(3-\sqrt{3}i)g}{36}, \\ a_{32}^1 = -\frac{g}{6}, \\ a_{34}^2 = \frac{(1+\sqrt{3}i)g}{12}, \\ a_{35}^2 = -\frac{\sqrt{3}ig}{18}, \\ a_{36}^2 = -\frac{(1-\sqrt{3}i)g}{24}, \end{cases} \tag{6.133}$$

$$\begin{cases} a_{41}^2 = \frac{g}{12}, \\ a_{42}^2 = -\frac{(3+\sqrt{3}i)g}{36}, \\ a_{43}^2 = \frac{(1+\sqrt{3}i)g}{12}, \\ a_{45}^2 = \frac{(1-\sqrt{3}i)g}{12}, \\ a_{46}^2 = -\frac{(3-\sqrt{3}i)g}{36}, \end{cases} \qquad \begin{cases} a_{51}^2 = \frac{(3+\sqrt{3}i)g}{36}, \\ a_{52}^2 = -\frac{(1+\sqrt{3}i)g}{24}, \\ a_{53}^2 = \frac{\sqrt{3}ig}{18}, \\ a_{54}^2 = \frac{(1-\sqrt{3}i)g}{12}, \\ a_{56}^2 = -\frac{g}{6}, \end{cases} \qquad \begin{cases} a_{61}^2 = \frac{(1+\sqrt{3}i)g}{12}, \\ a_{62}^2 = -\frac{\sqrt{3}ig}{18}, \\ a_{63}^2 = -\frac{(1-\sqrt{3}i)g}{24}, \\ a_{64}^2 = \frac{(3-\sqrt{3}i)g}{12}, \\ a_{65}^2 = -\frac{g}{6}. \end{cases} \tag{6.134}$$

It follows from (5.77) that

$$0 = \varepsilon_\nu^\top \Big( Q^{[2]'} + \frac{1}{6}\big(k_4 \Omega \varepsilon \varepsilon^\top \Omega^{-2} Q^{[1]} + k_3 \Omega \varepsilon \varepsilon^\top \Omega^{-3} Q^{[0]}\big)\Big)\varepsilon_\nu, \tag{6.135}$$

where $\nu = 1,2,3,4,5,6$, $k_4 = g$ and $k_3 = 2g'$. Let $K_1 = \frac{1}{6}k_4\Omega\varepsilon\varepsilon^\top\Omega^{-2}Q^{[1]}$ and $K_0 = \frac{1}{6}k_3\Omega\varepsilon\varepsilon^\top\Omega^{-3}Q^{[0]}$. Then it follows from (6.119), (6.123), (6.125), (6.121) and (6.129) that

$$K_1 = \begin{pmatrix} -\frac{gG}{36} & \frac{gG}{36} & -\frac{gG}{36} & \frac{gG}{36} & -\frac{gG}{36} & \frac{gG}{36} \\ -\frac{(1+\sqrt{3}i)gG}{72} & \frac{(1+\sqrt{3}i)gG}{72} & -\frac{(1+\sqrt{3}i)gG}{72} & \frac{(1+\sqrt{3}i)gG}{72} & -\frac{(1+\sqrt{3}i)gG}{72} & \frac{(1+\sqrt{3}i)gG}{72} \\ \frac{(1-\sqrt{3}i)gG}{72} & -\frac{(1-\sqrt{3}i)gG}{72} & \frac{(1-\sqrt{3}i)gG}{72} & -\frac{(1-\sqrt{3}i)gG}{72} & \frac{(1-\sqrt{3}i)gG}{72} & -\frac{(1-\sqrt{3}i)gG}{72} \\ \frac{gG}{36} & -\frac{gG}{36} & \frac{gG}{36} & -\frac{gG}{36} & \frac{gG}{36} & -\frac{gG}{36} \\ \frac{(1+\sqrt{3}i)gG}{72} & -\frac{(1+\sqrt{3}i)gG}{72} & \frac{(1+\sqrt{3}i)gG}{72} & -\frac{(1+\sqrt{3}i)gG}{72} & \frac{(1+\sqrt{3}i)gG}{72} & -\frac{(1+\sqrt{3}i)gG}{72} \\ -\frac{(1-\sqrt{3}i)gG}{72} & \frac{(1-\sqrt{3}i)gG}{72} & -\frac{(1-\sqrt{3}i)gG}{72} & \frac{(1-\sqrt{3}i)gG}{72} & -\frac{(1-\sqrt{3}i)gG}{72} & \frac{(1-\sqrt{3}i)gG}{72} \end{pmatrix}, \tag{6.136}$$

while it follows from (6.126), (6.117), (6.124) that

$$K_0 = \begin{pmatrix} \frac{g'}{3} & -\frac{g'}{3} & \frac{g'}{3} & -\frac{g'}{3} & \frac{g'}{3} & -\frac{g'}{3} \\ \frac{(1+\sqrt{3}i)g'}{6} & -\frac{(1+\sqrt{3}i)g'}{6} & \frac{(1+\sqrt{3}i)g'}{6} & -\frac{(1+\sqrt{3}i)g'}{6} & \frac{(1+\sqrt{3}i)g'}{6} & -\frac{(1+\sqrt{3}i)g'}{6} \\ -\frac{(1-\sqrt{3}i)g'}{6} & \frac{(1-\sqrt{3}i)g'}{6} & -\frac{(1-\sqrt{3}i)g'}{6} & \frac{(1-\sqrt{3}i)g'}{6} & -\frac{(1-\sqrt{3}i)g'}{6} & \frac{(1-\sqrt{3}i)g'}{6} \\ -\frac{g'}{3} & \frac{g'}{3} & -\frac{g'}{3} & \frac{g'}{3} & -\frac{g'}{3} & \frac{g'}{3} \\ -\frac{(1+\sqrt{3}i)g'}{6} & \frac{(1+\sqrt{3}i)g'}{6} & -\frac{(1+\sqrt{3}i)g'}{6} & \frac{(1+\sqrt{3}i)g'}{6} & -\frac{(1+\sqrt{3}i)g'}{6} & \frac{(1+\sqrt{3}i)g'}{6} \\ \frac{(1-\sqrt{3}i)g'}{6} & -\frac{(1-\sqrt{3}i)g'}{6} & \frac{(1-\sqrt{3}i)g'}{6} & -\frac{(1-\sqrt{3}i)g'}{6} & \frac{(1-\sqrt{3}i)g'}{6} & -\frac{(1-\sqrt{3}i)g'}{6} \end{pmatrix}. \tag{6.137}$$

Putting together (6.135), (6.136) and (6.137), we have $a_{11}^{2'} = -\frac{g'}{3} + \frac{gG}{36}$, $a_{22}^{2'} = \frac{(1+\sqrt{3}i)g'}{6} - \frac{(1+\sqrt{3}i)gG}{72}$, $a_{33}^{2'} = \frac{(1-\sqrt{3}i)g'}{6} - \frac{(1-\sqrt{3}i)gG}{72}$, $a_{44}^{2'} = -\frac{g'}{3} + \frac{gG}{36}$, $a_{55}^{2'} = \frac{(1+\sqrt{3}i)g'}{6} - \frac{(1+\sqrt{3}i)gG}{72}$ and $a_{66}^{2'} = \frac{(1-\sqrt{3}i)g'}{6} - \frac{(1-\sqrt{3}i)gG}{72}$. Thus

$$\begin{cases} a_{11}^2 = -\frac{g}{3} + \frac{G^2}{72}, \\ a_{22}^2 = \frac{(1+\sqrt{3}i)g}{6} - \frac{(1+\sqrt{3}i)G^2}{144}, \\ a_{33}^2 = \frac{(1-\sqrt{3}i)g}{6} - \frac{(1-\sqrt{3}i)G^2}{144}, \end{cases} \qquad \begin{cases} a_{44}^2 = -\frac{g}{3} + \frac{G^2}{72}, \\ a_{55}^2 = \frac{(1+\sqrt{3}i)g}{6} - \frac{(1+\sqrt{3}i)G^2}{144}, \\ a_{66}^2 = \frac{(1-\sqrt{3}i)g}{6} - \frac{(1-\sqrt{3}i)G^2}{144}. \end{cases} \tag{6.138}$$

It follows from (5.65) and (6.118) that

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1+\sqrt{3}i}{2} & -\frac{1-\sqrt{3}i}{2} & -1 & -\frac{1+\sqrt{3}i}{2} & \frac{1-\sqrt{3}i}{2} \\ 1 & -\frac{1-\sqrt{3}i}{2} & -\frac{1+\sqrt{3}i}{2} & 1 & -\frac{1-\sqrt{3}i}{2} & -\frac{1+\sqrt{3}i}{2} \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1+\sqrt{3}i}{2} & -\frac{1-\sqrt{3}i}{2} & 1 & -\frac{1+\sqrt{3}i}{2} & -\frac{1-\sqrt{3}i}{2} \\ 1 & \frac{1-\sqrt{3}i}{2} & -\frac{1+\sqrt{3}i}{2} & -1 & -\frac{1-\sqrt{3}i}{2} & \frac{1+\sqrt{3}i}{2} \end{pmatrix}. \tag{6.139}$$

Let $K = VQ^{[2]}$ and

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{pmatrix}.$$

As we want to find the function $\varphi_2$, then it follows from (5.82) that we need only to know $k_{11}$. Putting (6.133), (6.134), (6.137) and (6.138) together, we have

$$\begin{aligned} k_{11} &= a_{11}^2 + a_{21}^2 + a_{31}^2 + a_{41}^2 + a_{51}^2 + a_{61}^2 \\ &= \frac{G^2}{72} + \frac{1}{12}g. \end{aligned} \tag{6.140}$$

As $n_0 = 0$ and $k_5 = 0$, then it follows from (5.57) that

$$\varphi_0 = 1. \tag{6.141}$$

Using Proposition 5.28 it follows from (6.121), (6.129) and (6.140) that

$$\varphi_1 = -\frac{1}{6}G, \tag{6.142}$$

$$\varphi_2 = \frac{1}{72}G^2 + \frac{1}{12}g. \tag{6.143}$$

It follows from (5.58) the following result

**Proposition 6.31.** *Let $G(x) = \int_0^x g(t)dt$. Let $\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6\}$ be a canonical fundamen-*

*tal system of the differential equation* $-y^{(6)} - (gy'')'' = \lambda^2 y$. *Then*

$$\eta_\nu = \Big(1 - \frac{1}{6}e^{\frac{\pi i(-\nu+1)}{3}} G\mu^{-1} - e^{\frac{2\pi i(-\nu+1)}{3}} \Big(\frac{1}{72}G^2 + \frac{1}{12}g\Big)\mu^{-2}\Big) e^{e^{\frac{\pi i(\nu-1)}{3}}} + \{o(\mu^{-2})\}_\infty e^{e^{\frac{\pi i(\nu-1)}{3}}}, \tag{6.144}$$

*where* $\nu = 1, 2, 3, 4, 5, 6$.

## 6.8 Asymptotics of eigenvalues

The characteristic function for $g$ of the problem (6.1)–(6.7) is

$$D(\mu) = \det(\gamma_{j,k} \exp(\varepsilon_{j,k}))_{j,k=1}^{6}, \tag{6.145}$$

where $\varepsilon_{1,k} = \varepsilon_{2,k} = \varepsilon_{3,k} = 0$, $\varepsilon_{4,k} = \varepsilon_{5,k} = \varepsilon_{6,k} = \mu a e^{\frac{(k-1)i\pi}{3}}$, $\gamma_{1,k} = \delta_{k,0}(0,\mu)$, $\gamma_{2,k} = \delta_{k,1}(0,\mu)$, $\gamma_{3,k} = \delta_{k,2}(0,\mu)$, $\gamma_{4,k} = \delta_{k,0}(a,\mu)$, $\gamma_{5,k} = \delta_{k,2}(a,\mu)$, $\gamma_{6,k} = \delta_{k,4}(a,\mu) + \varepsilon\alpha\mu^3\delta_{k,1}(a,\mu)$, where $\gamma_{j,k}$, $j = 1, 2, 3, 4, 5, 6$ are given by the set of boundary conditions (6.2)–(6.7); with

$$\eta_\nu^{(j)}(x,\mu) = \delta_{\nu,j}(x,\mu) e^{e^{\frac{i\pi(\nu-1)}{3}}}, \tag{6.146}$$

$$\delta_{\nu,j}(x,\mu) = \Big[\frac{d^j}{dx^j}\Big]\Big\{\sum_{r=0}^{6}(\mu e^{\frac{i\pi(\nu-1)}{3}})^{-r}\varphi_r(x) e^{e^{\frac{i\pi(\nu-1)}{3}}}\Big\} e^{-e^{\frac{i\pi(\nu-1)}{3}}} + o(\mu^{-6+j}), \tag{6.147}$$

$j = 0, 1, 2, 3, 4, 5$, where $[\frac{d^j}{dx^j}]$ means that we omit those terms of Leibniz expansion which contains a function $\varphi_r^{(k)}$ with $k > 6 - r$ and where $\varphi_0$, $\varphi_1$ and $\varphi_2$ are as respectively defined in (6.141), (6.142) and (6.143), see (5.58).

In the same way as in (6.83) the characteristic function of the problem (6.1)–(6.7) can be written as

$$D(\mu) = \sum_{m=1}^{13} \psi_m(\mu) e^{\omega_m \mu a}, \tag{6.148}$$

where $\omega_m$, $m = 1\ldots, 13$ are as defined in (6.84).

**Remark 6.32.** Fix $\epsilon \in (0, \frac{\pi}{2a})$, for $k \in \mathbb{Z}$ let $R_{k,n}$ be the squares with vertices $\mu_{k,n} \pm \epsilon \pm i\epsilon$, where $\mu_{k,n}$ is as defined in (6.97). Let $y_j$, $j = 1, 2, 3, 4, 5, 6$ with $y_j^{(m)}(0) = \delta_{j,m+1}$ where

$m = 0, 1, 2, 3, 4, 5$, and $\delta$ is the Kronecker's delta, be a fundamental system for the differential equation (6.1). We denote $D_0$ the corresponding characteristic function $D$ for $g = 0$, we already know the asymptotic distribution of the eigenvalues for $g = 0$, see Proposition 6.28. The characteristic function of the differential equation (6.1) as defined in (2.36) is $\rho^6 + 1$. This characteristic function does not depend on the function $g$, also none of the boundary matrices of the eigenvalue problem depends of $g$. Hence the Birkhoff regularity of this problem is independent of $g$. Because of Birkoff regularity, $g$ influences lower order terms in $D$, thus the highest order terms do not vanish from $D$. Therefore it can be inferred that away from the small squares $R_{k,n}$ around the zeros of $D_0$, $|D(\mu) - D_0(\mu)| < |D_0(\mu)|$ if $|\mu|$ is sufficiently large. Since the fundamental system $y_j$, $j = 1, 2, 3, 4, 5, 6$, depends analytically on $\mu$, then $D$ and $D_0$ are analytic functions. Since $g$ influences only lower order terms in $D$, then it follows from Rouché's theorem that the eigenvalue problem (6.1)–(6.7) has for general $g$ the same $o(1)$ asymptotic distribution as for $g = 0$.

Hence Proposition 6.28 leads to the following theorem

**Theorem 6.33.** *For $g \in C^2[0, a]$ there exists a positive number $k_0$ such that the eigenvalues $\hat{\lambda}$, with sufficiently large modulus, of the problem* (6.1)–(6.7) *are*

$$\hat{\lambda}_{k,0} = \begin{cases} \frac{1}{8a^3}\left(2k\pi - i\ln\left|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\right| + o(1)\right)^3 \varepsilon \ \textit{if}\ \alpha > 3, \\ \frac{1}{8a^3}\left((2k+1)\pi - i\ln\left|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\right| + o(1)\right)^3 \varepsilon \ \textit{if}\ \alpha < 3, \end{cases}$$

*and*

$$\hat{\lambda}_{k,1} = \begin{cases} -\frac{1}{8a^3}\left(2k\pi - i\ln\left|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\right| + o(1)\right)^3 \varepsilon \ \textit{if}\ \alpha > 3, \\ -\frac{1}{8a^3}\left((2k+1)\pi - i\ln\left|\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\right| + o(1)\right)^3 \varepsilon \ \textit{if}\ \alpha < 3, \end{cases}$$

*where $s$ is any nonnegative integer, $|k| \geq k_0$, with $\varepsilon \in \{-1, 1\}$.*

It follows from (6.148) that

$$D_1(\mu) = D(\mu)e^{-2\mu a} = \psi_1(\mu) + \psi_2(\mu)e^{(-1+\sqrt{3}i)\mu a} + \sum_{m=3}^{13} \psi_m(\mu)e^{(\omega_m - \omega_1)\mu a}, \tag{6.149}$$

where $\omega_m - \omega_1$, $m = 3 \ldots, 13$ and $\omega_m - \omega_2$, $m = 3, \ldots, 13$ are as respectively defined in (6.87) and in (6.88). We recall that for $m = 3, \ldots, 13$, $\arg(\omega_m - \omega_1) \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$. It follows that for $m = 3 \ldots, 13$ and for $\arg\ \mu \in [-\frac{\pi}{12}, \frac{\pi}{12}]$, $\arg(\omega_m - \omega_1)\mu a \in [\frac{7\pi}{12}, \frac{17\pi}{12}] \subset (\frac{\pi}{2}, \frac{3\pi}{2})$. Therefore for

$m = 3, \ldots, 13$ and arg $\mu \in [-\frac{\pi}{12}, \frac{\pi}{12}]$, $|e^{(\omega_m - \omega_1)\mu a}| = e^{-\cos\frac{\pi}{12}|\mu| a} \leq e^{-\sin\frac{\pi}{12}|\mu| a}$ and the functions $\psi_m(\mu)e^{(\omega_m-\omega_1)\mu a}$ where $m = 3, \ldots, 13$ can be absorbed by $\widetilde{\psi}_1(\mu) = \psi_1(\mu) + o(\mu^{-s})$ for any nonnegative integer $s$. On the other hand, replacing $\omega_1$ by $\omega_2$ in (6.149) gives for $m = 3, \ldots, 13$ $\arg(\omega_m - \omega_2) \in [\pi, \frac{3\pi}{2}]$. Whence for arg $\mu \in [-\frac{\pi}{4}, -\frac{\pi}{12}]$, we have $\arg(\omega_m - \omega_2)\mu a \in [\frac{3\pi}{4}, \frac{17\pi}{12}] \subset (\frac{\pi}{2}, \frac{3\pi}{2})$ for $m = 3, \ldots, 13$. Thus for $m = 3, \ldots, 13$ and arg $\mu \in [-\frac{\pi}{4}, -\frac{\pi}{12}]$, $|e^{(\omega_m-\omega_2)\mu a}| = e^{-\cos\frac{\pi}{4}|\mu|a} < e^{-\sin\frac{\pi}{12}|\mu|a}$ and the functions $\psi_m(\mu)e^{(\omega_m-\omega_2)\mu a}$ where $m = 3, \ldots, 13$ can be absorbed by $\widetilde{\psi}_2(\mu) = \psi_2(\mu) + o(\mu^{-s})$ for any nonnegative integer $s$. Thus in the sector $[-\frac{\pi}{4}, \frac{\pi}{12}]$,

$$\widetilde{D}_1(\mu) = \widetilde{\psi}_1(\mu) + \widetilde{\psi}_2(\mu)e^{(-1+\sqrt{3}i)\mu a}, \tag{6.150}$$

where

$$\widetilde{\psi}_1(\mu) = \begin{vmatrix} \gamma_{4,1} & \gamma_{4,2} & \gamma_{4,6} \\ \gamma_{5,1} & \gamma_{5,2} & \gamma_{5,6} \\ \gamma_{6,1} & \gamma_{6,2} & \gamma_{6,6} \end{vmatrix} \cdot \begin{vmatrix} \gamma_{1,3} & \gamma_{1,4} & \gamma_{1,5} \\ \gamma_{2,3} & \gamma_{2,4} & \gamma_{2,5} \\ \gamma_{3,3} & \gamma_{3,4} & \gamma_{3,5} \end{vmatrix} \tag{6.151}$$

and

$$\widetilde{\psi}_2(\mu) = -\begin{vmatrix} \gamma_{4,1} & \gamma_{4,2} & \gamma_{4,3} \\ \gamma_{5,1} & \gamma_{5,2} & \gamma_{5,3} \\ \gamma_{6,1} & \gamma_{6,2} & \gamma_{6,3} \end{vmatrix} \cdot \begin{vmatrix} \gamma_{1,4} & \gamma_{1,5} & \gamma_{1,6} \\ \gamma_{2,4} & \gamma_{2,5} & \gamma_{2,6} \\ \gamma_{3,4} & \gamma_{3,5} & \gamma_{3,6} \end{vmatrix}. \tag{6.152}$$

Let

$$\widetilde{\psi}_{11}(\mu) = \begin{vmatrix} \gamma_{4,1} & \gamma_{4,2} & \gamma_{4,6} \\ \gamma_{5,1} & \gamma_{5,2} & \gamma_{5,6} \\ \gamma_{6,1} & \gamma_{6,2} & \gamma_{6,6} \end{vmatrix}, \quad \widetilde{\psi}_{12}(\mu) = \begin{vmatrix} \gamma_{1,3} & \gamma_{1,4} & \gamma_{1,5} \\ \gamma_{2,3} & \gamma_{2,4} & \gamma_{2,5} \\ \gamma_{3,3} & \gamma_{3,4} & \gamma_{3,5} \end{vmatrix}, \tag{6.153}$$

and

$$\widetilde{\psi}_{21}(\mu) = \begin{vmatrix} \gamma_{4,1} & \gamma_{4,2} & \gamma_{4,3} \\ \gamma_{5,1} & \gamma_{5,2} & \gamma_{5,3} \\ \gamma_{6,1} & \gamma_{6,2} & \gamma_{6,3} \end{vmatrix}, \quad \widetilde{\psi}_{22}(\mu) = -\begin{vmatrix} \gamma_{1,4} & \gamma_{1,5} & \gamma_{1,6} \\ \gamma_{2,4} & \gamma_{2,5} & \gamma_{2,6} \\ \gamma_{3,4} & \gamma_{3,5} & \gamma_{3,6} \end{vmatrix}. \tag{6.154}$$

The computations of the functions $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$ are done respectively by the lines of code "The computation of psi1 starts here" and "The computation of psi2 starts here" of Subsection 7.5.2. Inside the first lines of code can be found the computation of $\widetilde{\psi}_{11}$ as well as the computation of $\widetilde{\psi}_{12}$. While inside "The computation of psi2 starts here" can be found the computation of $\widetilde{\psi}_{21}$ as well as the computation of $\widetilde{\psi}_{22}$. It follows from (6.153) that

$$\begin{aligned}\widetilde{\psi}_{11}(\mu) &= \sqrt{3}i(\varepsilon\alpha-3)\mu^6 + 2\sqrt{3}i(\varepsilon\alpha-3)\varphi_1(a)\mu^5 + 2\sqrt{3}i((\varepsilon\alpha-3)\varphi_1^2(a) \\ &\quad - \varepsilon\alpha\varphi_1'(a))\mu^4 + o(\mu^4), \qquad (6.155) \\ \widetilde{\psi}_{12}(\mu) &= \sqrt{3}i\mu^3 - 2\sqrt{3}i\varphi_1'(0)\mu + o(\mu), \qquad (6.156)\end{aligned}$$

while (6.154) gives

$$\begin{aligned}\widetilde{\psi}_{21}(\mu) &= -\sqrt{3}i(\varepsilon\alpha+3)\mu^6 - (3+\sqrt{3}i)(\varepsilon\alpha+3)\varphi_1(a)\mu^5 \\ &\quad - (3-\sqrt{3}i)((\varepsilon\alpha+3)\varphi_1^2(a) - \varepsilon\alpha\varphi_1'(a))\mu^4 + o(\mu^4), \qquad (6.157) \\ \widetilde{\psi}_{22}(\mu) &= \sqrt{3}i\mu^3 - (3-\sqrt{3}i)\varphi_1'(0)\mu + o(\mu). \qquad (6.158)\end{aligned}$$

Recall that $\varphi_1(x) = -\frac{1}{6}\int_0^x g(t)dt$, see (6.142). Hence it follows from (6.128), (6.155) and (6.156) that

$$\begin{aligned}\widetilde{\psi}_1(\mu) &= -3(\varepsilon\alpha-3)\mu^9 - 6(\varepsilon\alpha-3)\varphi_1(a)\mu^8 - 6\big((\varepsilon\alpha-3)\varphi_1^2(a) - \varepsilon\alpha\varphi_1'(a) \\ &\quad - (\varepsilon\alpha-3)\varphi_1'(0)\big)\mu^7 + o(\mu^7), \qquad (6.159) \\ \widetilde{\psi}_2(\mu) &= 3(\varepsilon\alpha+3)\mu^9 + 3(1-\sqrt{3}i)(\varepsilon\alpha+3)\varphi_1(a)\mu^8 - 3(1+\sqrt{3}i) \\ &\quad \times \big((\varepsilon\alpha+3)\varphi_1^2(a) - (\varepsilon\alpha+3)\varphi_1'(0) - \varepsilon\alpha\varphi_1'(a)\big)\mu^7 + o(\mu^7). \qquad (6.160)\end{aligned}$$

The zeros $\hat{\mu}_{k,n}$ of $D$ satisfy the asymptotics $\hat{\mu}_{k,n} = (-1)^n\big(-\frac{2(1+\sqrt{3}i)ki\pi}{4a} + \tau_0\big)e^{\frac{n\pi i}{3}} + o(1)$ as $k \to \infty$, see Theorem 6.33. Let

$$\hat{\mu}_k = -\frac{(1+\sqrt{3}i)ki\pi}{2a} + \tau(k),\ \tau(k) = \sum_{m=0}^{s}\tau_m k^{-m} + o(k^{-s}),\ k = 1, 2, \ldots, \qquad (6.161)$$

where

$$\tau_0 = \begin{cases} -\frac{1+\sqrt{3}i}{4a}\ln\big(\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\big) \text{ if } \alpha > 3, \\ -\frac{1+\sqrt{3}i}{4a}\big(\ln\big(\frac{3-\varepsilon\alpha}{\varepsilon\alpha+3}\big) + i\pi\big) \text{ if } \alpha < 3, \end{cases} \qquad (6.162)$$

be the asymptotics of the zeros of $D$ in the sector $[-\frac{\pi}{4}, \frac{\pi}{12}]$. Hence

$$e^{(-1+\sqrt{3}i)\tau_0 a} = \frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}, \qquad (6.163)$$

where $\alpha \neq 3$. It follows from the Taylor expansions of the function $x \mapsto e^x$ and $x \mapsto \frac{1}{1+x}$ that

$$\begin{aligned}
e^{(-1+\sqrt{3}i)\hat{\mu}_k a} &= e^{2ki\pi} e^{(-1+\sqrt{3}i)\tau_0 a} \exp\left((-1+\sqrt{3}i)a\left(\frac{\tau_1}{k}+\frac{\tau_2}{k^2}+o(k^{-2})\right)\right)\\
&= e^{(-1+\sqrt{3}i)\tau_0 a}\Big(1+(-1+\sqrt{3}i)a\frac{\tau_1}{k} - ((1+\sqrt{3}i)a^2\tau_1^2\\
&+ (1-\sqrt{3}i)\tau_2 a)\frac{1}{k^2}\Big) + o(k^{-2})\\
&= \Big(\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big)\Big(1+(-1+\sqrt{3}i)a\frac{\tau_1}{k} - ((1+\sqrt{3}i)a^2\tau_1^2\\
&+ (1-\sqrt{3}i)\tau_2 a)\frac{1}{k^2}\Big) + o(k^{-2}),
\end{aligned} \tag{6.164}$$

and

$$\frac{1}{\hat{\mu}_k} = \frac{(\sqrt{3}+i)a}{2k\pi}\Big(1+\frac{(\sqrt{3}+i)a\tau(k)}{2k\pi}\Big)^{-1} = \frac{(\sqrt{3}+i)a}{2k\pi} - \frac{(1+\sqrt{3}i)a^2\tau_0}{2k^2\pi^2} + o(k^{-2}). \tag{6.165}$$

We know that $\widetilde{D}_1(\hat{\mu}_k)=0$ can be written as

$$\hat{\mu}_k^{-9}\widetilde{\psi}_1(\hat{\mu}_k) + \hat{\mu}_k^{-9}\widetilde{\psi}_2(\hat{\mu}_k)e^{(-1+\sqrt{3}i)\tau(k)a} = 0. \tag{6.166}$$

Substituting (6.164) and (6.165) in (6.166), we get from (6.159) and (6.160)

$$\begin{aligned}
&-3(\varepsilon\alpha-3) - 3(\varepsilon\alpha-3)\varphi_1(a)\Big(\frac{(\sqrt{3}+i)a}{k\pi} - \frac{(1+\sqrt{3}i)a^2\tau_0}{k^2\pi^2}\Big)\\
&-\frac{3(1+\sqrt{3}i)a^2}{k^2\pi^2}\big((\varepsilon\alpha-3)\varphi_1^2(a) - \varepsilon\alpha\varphi_1'(a) - (\varepsilon\alpha-3)\varphi_1'(0)\big) + \Big(\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big)\\
&\times\Big(3(\varepsilon\alpha+3) + 3(1-\sqrt{3}i)(\varepsilon\alpha+3)\varphi_1(a)\Big(\frac{(\sqrt{3}+i)a}{2k\pi} - \frac{(1+\sqrt{3}i)a^2\tau_0}{2k^2\pi^2}\Big)\\
&+\frac{3(1-\sqrt{3}i)a^2}{k^2\pi^2}\big((\varepsilon\alpha+3)\varphi_1^2(a) - (\varepsilon\alpha+3)\varphi_1'(0) - \varepsilon\alpha\varphi_1'(a)\big)\Big) + \frac{(-1+\sqrt{3}i)a\tau_1}{k}\\
&\times\Big(\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big)\Big(3(\varepsilon\alpha+3) + 3(1-\sqrt{3}i)(\varepsilon\alpha+3)\varphi_1(a)\Big(\frac{(\sqrt{3}+i)a}{2k\pi} - \frac{(1+\sqrt{3}i)a^2\tau_0}{2k^2\pi^2}\Big)\\
&+\frac{3(1-\sqrt{3}i)a^2}{k^2\pi^2}\big((\varepsilon\alpha+3)\varphi_1^2(a) - (\varepsilon\alpha+3)\varphi_1'(0) - \varepsilon\alpha\varphi_1'(a)\big)\Big) - \Big(\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\Big)\\
&\times\frac{1}{k^2}\big((1+\sqrt{3}i)a^2\tau_1^2 + (1-\sqrt{3}i)a\tau_2\big)\Big(3(\varepsilon\alpha+3) + 3(1-\sqrt{3}i)(\varepsilon\alpha+3)\varphi_1(a)\\
&\times\Big(\frac{(\sqrt{3}+i)a}{2k\pi} - \frac{(1+\sqrt{3}i)a^2\tau_0}{2k^2\pi^2}\Big).
\end{aligned} \tag{6.167}$$

Comparing the coefficients of $k^{-1}$ and $k^{-2}$ in (6.167), we get

$$\tau_1 = \frac{\sqrt{3}-i}{2\pi}\varphi_1(a) = -\frac{1+\sqrt{3}i}{4a}\Big(\frac{2ia}{\pi}\varphi_1(a)\Big), \tag{6.168}$$

$$\begin{aligned}\tau_2 &= a\Big(\frac{\tau_0\varphi_1(a)}{\pi^2} - \frac{(\sqrt{3}-i)\tau_1\varphi_1(a)}{\pi} + \frac{1-\sqrt{3}i}{2}\tau_1^2\Big) + \frac{(3-\sqrt{3}i)a}{2\pi^2}(\varphi_1^2(a) - \varphi_1'(0))\\ &+ \frac{(1+\sqrt{3}i)(3+\sqrt{3}\varepsilon\alpha i)a\varepsilon\alpha\varphi_1'(a)}{2\pi^2(\varepsilon^2\alpha^2-9)}\\ &= -\frac{1+\sqrt{3}i}{4a}\Big[-(1-\sqrt{3}i)a^2\Big(\frac{\tau_0\varphi_1(a)}{\pi^2} - \frac{(\sqrt{3}-i)\tau_1\varphi_1(a)}{\pi} + \frac{1-\sqrt{3}i}{2}\tau_1^2\Big)\\ &+ \frac{2\sqrt{3}ia^2}{\pi^2}(\varphi_1^2(a) - \varphi_1'(0)) - \frac{(3+\sqrt{3}i\varepsilon\alpha)a^2\varepsilon\alpha\varphi_1'(a)}{\pi^2(\varepsilon^2\alpha^2-9)}\Big].\end{aligned} \tag{6.169}$$

We recall that $\varphi_1(x) = -\frac{1}{6}G(x) = -\frac{1}{6}\int_0^x g(t)dt$. Thus

$$\tau_1 = -\frac{1+\sqrt{3}i}{4a}\Big(-\frac{ia}{3\pi}G(a)\Big), \tag{6.170}$$

$$\begin{aligned}\tau_2 &= -\frac{(1+\sqrt{3}i)}{4a}\Big[\frac{(1-\sqrt{3}i)a^2}{6\pi^2}\Big(G(a)\tau_0 - (1-\sqrt{3}i)G(a) + \frac{1+\sqrt{3}i}{12}G^2(a)\Big)\\ &+ \frac{\sqrt{3}ia^2}{\pi^2}\Big(\frac{1}{18}G^2(a) + \frac{1}{3}g(0)\Big) + \frac{\varepsilon\alpha a^2}{\varepsilon\alpha^2-9}\,\frac{3+\sqrt{3}i\varepsilon\alpha}{6}\,\frac{g(a)}{\pi^2}\Big].\end{aligned} \tag{6.171}$$

Recall that

$$\tau_0 = \begin{cases} -\frac{1+\sqrt{3}i}{4a}\ln\left(\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\right) \text{ if } \alpha > 3,\\ -\frac{1+\sqrt{3}i}{4a}\Big(\ln\left(\frac{3-\varepsilon\alpha}{\varepsilon\alpha+3}\right) + i\pi\Big) \text{ if } \alpha < 3,\end{cases}$$

see (6.162).

Let

$$\widetilde{\tau}_0 = \begin{cases} \ln\left(\frac{\varepsilon\alpha-3}{\varepsilon\alpha+3}\right) \text{ if } \alpha > 3,\\ \Big(\ln\left(\frac{3-\varepsilon\alpha}{\varepsilon\alpha+3}\right) + i\pi\Big) \text{ if } \alpha < 3\end{cases} \tag{6.172}$$

and

$$\widetilde{\tau}_1 = \Big(-\frac{ia}{3\pi}G(a)\Big), \tag{6.173}$$

$$\begin{aligned}\widetilde{\tau}_2 &= \Big[\frac{(1-\sqrt{3}i)a^2}{6\pi^2}\Big(G(a)\tau_0 - (1-\sqrt{3}i)G(a) + \frac{1+\sqrt{3}i}{12}G^2(a)\Big)\\ &+ \frac{\sqrt{3}ia^2}{\pi^2}\Big(\frac{1}{18}G^2(a) + \frac{1}{3}g(0)\Big) + \frac{\varepsilon\alpha a^2}{\varepsilon\alpha^2-9}\,\frac{3+\sqrt{3}i\varepsilon\alpha}{6}\,\frac{g(a)}{\pi^2}\Big].\end{aligned} \tag{6.174}$$

Hence we have from Remark 6.27 the following result,

**Theorem 6.34.** *Let $\varepsilon \in \{-1,1\}$, $\alpha$ be a nonnegative number and $\alpha \neq 3$. Then for $g \in C^2[0,a]$, there exists a positive number $k_0$ such that the eigenvalues $\hat{\lambda}$, with sufficiently large modulus, of the problem* (6.1)–(6.7) *are*

$$\hat{\lambda}_{k,0} = \frac{1}{8a^3}\Big(2k\pi - i\Big(\widetilde{\tau}_0 + \frac{\widetilde{\tau}_1}{k} + \frac{\widetilde{\tau}_2}{k^2}\Big) + o(k^{-2})\Big)^3 \varepsilon$$

*and*

$$\hat{\lambda}_{k,1} = -\frac{1}{8a^3}\Big(2k\pi - i\Big(\widetilde{\tau}_0 + \frac{\widetilde{\tau}_1}{k} + \frac{\widetilde{\tau}_2}{k^2}\Big) + o(k^{-2})\Big)^3 \varepsilon,$$

*where $\widetilde{\tau}_0$, $\widetilde{\tau}_1$ and $\widetilde{\tau}_2$ are as respectively defined in* (6.172), (6.173) *and* (6.174), *and where $|k| \geq k_0$ and $\varepsilon \in \{-1,1\}$.*

# Chapter 7

# Appendix

## 7.1 Introduction

We present in this appendix, for the reader who is interested, the notion and properties of convex hull in Section 7.2, the definition and properties of estimates of exponential sums in Section 7.3 and finally the Sage's codes that we have used to compute the first four terms of the eigenvalue asymptotics given respectively in Chapter 5 and Chapter 6.

## 7.2 The convex hull of sums of complex numbers

**Definition 7.1.** A set $G$ is convex if given two points $a$ and $b$ in $G$ the line segment joining $a$ and $b$ , $[a, b]$, lies entirely in $G$. The set $G$ is star shaped if there is a point $a$ in $G$ such that for each $z$ in $G$, the line segment $[a, z]$ lies entirely in $G$. See Definition 4.3. [12, page 82].

**Remark 7.2.** [12, page 82] Each convex set is star shaped but the converse is false.

**Definition 7.3.** [44, page 269]. For any set $A \subset \mathbb{C}$ the intersection of all convex sets containing $A$ is called the convex hull of $A$ and denoted $\operatorname{conv} A$.

**Proposition 7.4.** [44, page 269].

$$\operatorname{conv}\{c_1, \dots, c_n\} = \left\{ z \in \mathbb{C} : z = \sum_{\nu=1}^{n} \lambda_\nu c_\nu;\ \lambda_1 \geq 0, \dots, \lambda_n \geq 0,\ \sum_{\nu=1}^{n} \lambda_\nu = 1 \right\}.$$

**General Assumptions 7.5.** [31, page 441]. Let $n \in \mathbb{N}\backslash\{0\}$. We consider the sets $\mathscr{P}_1, \dots, \mathscr{P}_n \subset \mathbb{C}$ with the property that, for each $j \in \{1, \dots, n\}$, $0 \in \mathscr{P}_j$ such that $c_j \neq 0$ and $\mathscr{P}_j \subset \overline{0, c_j}$. We set

$$\mathscr{E} := \left\{ \sum_{j=1}^{n} z_j : z_j \in \mathscr{P}_j,\ j = 1, \dots, n \right\}. \tag{7.1}$$

Let $\mathscr{P}$ be the convex hull of $\mathscr{E}$.

There is a natural number $m$ with $1 \leq m \leq n$ such that the points of $\mathscr{P}_1, \dots, \mathscr{P}_n$ lie on $m$ different lines $g_1, \dots, g_m$ with $0 \in g_j$ $(j = 1, \dots, m)$. We have $g_j = \mathbb{R}e^{i\phi_j}$, where we may assume without loss of generality that $0 \leq \phi_1 < \phi_2 < \dots < \phi_m < \pi$.

For $l \in \mathbb{Z}\backslash\{0\}$ and $j \in \{1, \dots, m\}$ we set $\phi_{lm+j} := \phi_j + \pi$ and $g_{lm+j} := g_j$. Then $\phi_k < \phi_{k+1}$ and $\phi_{k+m} = \phi_k + \pi$ hold for all $k \in \mathbb{Z}$. For $j \in \mathbb{Z}$ we set

$$a_j := \sum_{k=1,\ c_k \in \mathbb{R}_+ e^{i\pi_j}}^{n} c_k$$

and

$$\mathscr{E}^j := \left\{ \sum_{k=1,\ c_k \in g_j}^{n} z_k : z_k \in \mathscr{P}_k,\ k = 1, \dots, n \right\}.$$

**Proposition 7.6.** *For each $j \in \mathbb{Z}$ we have $\mathscr{E}^j \subset \mathscr{E}$, and the line segment $\overline{a_j, a_{j+m}}$ is the convex hull of $\mathscr{E}^j$. Let $z_k \in \overline{0, c_k}$ $(k = 1, \dots, n;\ c_k \in g_j)$ and*

$$\sum_{k=1,\ c_k \in g_j} z_k = a_j.$$

*Then*

$$z_k = \begin{cases} c_k & if\ c_k \in \mathbb{R}_+ e^{i\phi_j}, \\ 0 & if\ c_k \in \mathbb{R}_- e^{i\phi_j}. \end{cases} \tag{7.2}$$

See Proposition A.1.1. [31, page 442].

**Proposition 7.7.** *The convex hull $\mathscr{P}$ of $\mathscr{E}$ is*

$$\mathscr{P} = \sum_{j=1}^{m} \overline{a_j, a_{m+j}}.$$

See Proposition A.1.2. [31, page 442].

**Remark 7.8.** [31, page 443]. For $j \in \mathbb{Z}$ we set

$$b_j := \sum_{k=j}^{j+m-1} a_k \in \mathscr{E}.$$

Note that $b_{j+1} - b_j = a_{j+m} - a_j \neq 0$. The definition of $a_k$ yields

$$b_j = \sum_{k=j}^{j+m-1} \sum_{\nu=1,\ c_\nu \in \mathbb{R}_+ e^{i\phi_k}} c_v.$$

Since $c_\nu \in \mathbb{R}_+ e^{i\phi_k}$ holds for some $k \in \{j, \dots, j+m-1\}$ if and only if $\arg c_\nu - \phi_j \in [0, \pi)$ mod $(\pi)$, we obtain the representation

$$b_j = \sum_{\nu=1,\ \phi_j \leq \arg c_\nu < \phi_j + \pi}^{n} c_\nu. \tag{7.3}$$

**Theorem 7.9.** i) *$\mathscr{P}$ is a convex polygon with $2m$ vertices, the set of vertices of $\mathscr{P}$ is*

$$\widetilde{\mathscr{E}} = \{b_j : j = 1, \dots, 2m\},$$

*and the boundary of $\mathscr{P}$ is*

$$\partial \mathscr{P} = \bigcup_{j=1}^{2m} \overline{b_j, b_{j+1}}.$$

ii) *Let $j \in \{1, \dots, 2m\}$, $z_k \in \overline{0, c_k}$ $(k = 1, \dots, n)$ and $z = \sum_{k=1}^{n} z_k$. Then $z \in \overline{b_j, b_{j+1}}$ if and only if for all $l \in \{j+1, \dots, j+m-1\}$ and all $k \in \{1, \dots, n\}$ with $c_k \in g_l$*

$$z_k = \begin{cases} c_k & if\ c_k \in \mathbb{R}_+ e^{i\phi_l}, \\ 0 & if c_k \in \mathbb{R}_- e^{i\phi_l}. \end{cases}$$

iii) *The representation of $b_j$ $(j = 1, \dots, 2m)$ as an element of $\mathscr{E}$ is unique.*

See Theorem A.1.3. [31, page 443].

## 7.3 Estimates of exponential sums

Let $\Omega$ be an unbounded subset of $\mathbb{C}$. We may assume for simplicity that $|\lambda| \geq 1$ for $\lambda \in \Omega$. For $\widetilde{a} : \Omega \to \mathbb{C}$ and $a \in \mathbb{C}$, we write

$$\widetilde{a}(\lambda) = [a] \ \text{ if } \widetilde{a}(\lambda) - a \to 0 \quad \text{as} \quad \lambda \to \infty.$$

For $\nu \in \mathbb{R}$ and $\lambda \in \mathbb{C}\backslash\{0\}$ let $\lambda^\nu = \exp(\nu \log \lambda)$, where log is the principal value of logarithm, i.e. the inverse of $\exp : \mathbb{R} + i(-\pi, \pi] \to \mathbb{C}\backslash\{0\}$. For $\nu \in \mathbb{N}$, $\lambda^\nu$ is the $\nu$-th power of lambda.

**Proposition 7.10.** [31, page 450]. *For $\lambda \in \mathbb{C}\backslash\{0\}$ we have*

$$\arg(-\lambda) := \Im(\log(-\lambda)) = \begin{cases} \pi + \arg\lambda & \textit{if } \arg\lambda \leq 0, \\ -\pi + \arg\lambda & \textit{if } \arg\lambda > 0. \end{cases}$$

*Hence*

$$(-\lambda)^\nu = (-1)^\nu \lambda^\nu \qquad \textit{if } \arg\lambda \leq 0,$$
$$(-\lambda)^\nu = (-1)^{-\nu} \lambda^\nu \qquad \textit{if } \arg\lambda > 0.$$

**Definition 7.11.** [31, page 450]. Let $\mathscr{N}$ be a countable set with at least two elements $c_j \in \mathbb{C}$ be pairwise different and $b_j : \Omega \to \mathbb{C}$ for $j \in \mathscr{N}$. Suppose that

$$\sup\{|c_j| : j \in \mathscr{N}\} < \infty. \tag{7.4}$$

We may assume that for all $j \in \mathscr{N}$ there are $a_j \in \mathbb{C}$ and $\nu_j \in \mathbb{R}$ such that

$$b_j(\lambda) = \lambda^{\nu_j}[a_j], \qquad \sup\{\nu_j : j \in \mathscr{N}\} < \infty, \tag{7.5}$$

$$\sum_{j\in\mathscr{N}} |a_j| < \infty, \tag{7.6}$$

and

$$\varepsilon(\lambda) := \sum_{j\in\mathscr{N}} |\varepsilon_j(\lambda)| < \infty, \qquad \varepsilon(\lambda) \to 0 \qquad (\lambda \to \infty), \tag{7.7}$$

where

$$\varepsilon_j(\lambda) := \lambda^{-\nu_j} b_j(\lambda) - a_j \qquad (j \in \mathscr{N}). \tag{7.8}$$

The function $D : \Omega \to \mathbb{C}$ defined by

$$D(\lambda) := \sum_{j\in\mathscr{N}} b_j(\lambda) \exp(c_j\lambda) \qquad (\lambda \in \Omega) \tag{7.9}$$

is called the exponential sum.

**Remark 7.12.** [31, page 451]. The estimate $|b_j(\lambda)\exp(c_j\lambda)| \leq |\lambda|^{\nu_j}(|a_j| + |\varepsilon_j(\lambda)|)\exp\{|c_j\lambda|\}$, the boundedness of the set $c_j$ and the set $\nu_j$, the assumptions (7.6) and (7.7) prove that the exponential sum (7.9) is absolutely convergent.

**Proposition 7.13.** *We consider the exponential sum* (7.9)

$$D(\lambda) = \sum_{j \in \mathscr{N}} b_j(\lambda) \exp(c_j \lambda),$$

*where* $c_j \in \mathbb{R}$ $(j \in \mathscr{N})$*. Assume that there are* $\alpha, \beta \in \mathscr{N}$ *such that* $c_\alpha \leq 0$*,* $c_\beta \geq 0$*,* $c_\alpha < c_j < c_\beta$ $(j \in \mathscr{N} \backslash \{\alpha, \beta\})$*,* $\alpha \neq 0$*,* $\beta \neq 0$*. Assume that there is a number* $d \in \mathbb{R}$ *such that* $\nu_j = dc_j$ *for all* $j \in \mathscr{N}$*. Then there are numbers* $M > 0$*,* $K_0 \geq 0$ *and* $g_0 > 0$ *such that for all* $\lambda \in \Omega$ *satisfying* $|\lambda| > K_0$ *and* $|\Re(\lambda) + d \log |\lambda|| \geq M$ *the estimate*

$$|D(\lambda)| \geq g_0$$

*holds.* See Proposition A.2.3.[31, page 453].

**Proposition 7.14.** *We consider the exponential sum* (7.9)

$$D(\lambda) = \sum_{j \in \mathscr{N}} b_j(\lambda) \exp(c_j \lambda),$$

*where* $b_j(\lambda) = \lambda^{\nu_j}[a_j]$ *according to* (7.5)–(7.8)*,* $\nu_j \in \mathbb{R}$*,* $c_j \in \mathbb{R}$ $(j \in \mathscr{N})$*,* $\alpha, \beta \in \mathscr{N}$*,* $a_\alpha \neq 0$*,* $a_\beta \neq 0$*,* $0 = c_\alpha < c_j < c_\beta$ $(j \in \mathscr{N} \backslash \{\alpha, \beta\})$*. Assume that there is a number* $d \in \mathbb{R}$ *such that* $\nu_j = dc_j$ *for all* $j \in \mathscr{N}$*. Then there are a natural number* $l$*, for each* $\delta > 0$ *numbers* $K(\delta)$ *and* $g(\delta) > 0$*, and for all* $R > K(\delta)$ *there are* $l$ *balls of radius* $\delta$ *such that for all* $\lambda \in \Omega$ *with* $R \leq |\lambda| \leq R + 1$ *outside of these balls the estimate*

$$|D(\lambda)| \geq g(\delta)$$

*holds.* See Proposition A.2.6. [31, page 457].

**Definition 7.15.** We consider an exponential sum of the form (7.9) with the representation (7.5) of $b_j$. Let $\mathscr{M}, \mathscr{M}' \subset \mathscr{N}$, where $\#\mathscr{M} \geq 2$ and $\mathscr{M}'$ is a finite subset of $\mathscr{M}$. The pair $(\mathscr{M}, \mathscr{M}')$ is called weakly regular if the following properties hold

i) there are $\alpha, \beta \in \mathscr{M}$ such that for all $j \in \mathscr{M}$ there is a number $\tau \in [0, 1]$ such that $c_j = \tau c_\alpha + (1 - \tau) c_\beta$, i.e., the line segment $\overline{c_\alpha, c_\beta}$ is the convex hull of the set $\{c_j : j \in \mathscr{M}\}$;

ii) for all $j \in \mathscr{M}$ there are $j_1, j_2 \in \mathscr{M}'$ and a number $\tau \in [0, 1]$ such that $c_j = \tau c_{j_1} + (1 - \tau) c_{j_2}$ and $\nu_j \leq \tau \nu_{j_1} + (1 - \tau) \nu_{j_2}$;

iii) if $j \in \mathscr{M}'$ for a triple $\{j, j_1, j_2\}$ which fulfills ii), then $j \in \{j_1, j_2\}$.

See Definition A.2.7. [31, page 461].

**Remark 7.16.** Assume that $\mathscr{M}$ is finite and the points $\{c_j : j \in \mathscr{M}\}$ lie on a straight line. Then there is a subset $\mathscr{M}'$ of $\mathscr{M}$ such that the pair $(\mathscr{M}, \mathscr{M}')$ is weakly regular. See Remark A.2.9. [31, page 469].

**Remark 7.17.** Let $D(\lambda)$ be an exponential sum with the representation (7.5) of $b_j$. Let $\mathscr{M} \subset \mathscr{N}$ be such that the convex hull of $\mathscr{M}$ is a line segment, the endpoints of which we denote by $c_\alpha$ and $c_\beta$. Assume that on $\mathscr{M}$ the $\nu_j$ do not depend on $j$, i.e., there is a $\nu \in \mathbb{R}$ such that $\nu_j = \nu$ for all $j \in \mathscr{M}$. Then $(\mathscr{M}, \{\alpha, \beta\})$ is weakly regular. See Remark A.2.10. [31, page 463].

**Proposition 7.18.** *We consider the exponential sum* (7.9)

$$D(\lambda) = \sum_{j \in \mathscr{N}} b_j(\lambda) \exp(c_j \lambda),$$

*where $b_j(\lambda) = \lambda^{\nu_j}[a_j]$ according to (7.5)–(7.8), $\nu_j \in \mathbb{R}$, $c_j \in \mathbb{R}_+$ $(j \in \mathscr{N})$, $c_{j_0} = 0$, $\nu_{j_0} = 0$ for some $j_0 \in \mathscr{N}$. Assume that there is a finite subset $\mathscr{N}'$ of $\mathscr{N}$ such that $(\mathscr{N}', \mathscr{N})$ is weakly regular and $a_j \neq 0$ for all $j \in \mathscr{N}'$. Then there are a natural number $l > 0$, for each $\delta > 0$ numbers $K(\delta) > 0$ and $g(\delta) > 0$ and for all $R > K(\delta)$ there are $l$ balls of radius $\delta$ such that for all $\lambda \in \Omega$ with $R \leq |\lambda| \leq R + 1$ outside these balls we have the estimates*

$$|D(\lambda)| \geq g(\delta).$$

See Proposition A.2.11. [31, page 463].

**Proposition 7.19.** [31, page 468]. *We consider a general exponential sum* (7.9) *and set*

$$\mathscr{E} := \{c_j : j \in \mathscr{N}\}.$$

*Let $\mathscr{P}$ be the convex hull of $\mathscr{E}$. If $\mathscr{P}$ is a convex polygon, then there are a number $S$ and line segments $P_s$ $(s = 1, \dots, S)$ such that $\partial \mathscr{P} = \bigcup\limits_{s=1}^{S} P_s$, where the endpoints of $P_s$ are the vertices of $\mathscr{P}$.*

**Definition 7.20.** Let $\mathscr{N}_s := \{j \in \mathscr{N} : c_j \in P_s\}$, $(s = 1, \ldots, S)$. The exponential sum (7.9) with the representation (7.5) of $b_j$ is called weakly regular if the following three conditions hold:

i) $\mathscr{P}$ is a convex polygon;

ii) for each $s \in \{1, \ldots, S\}$ there is finite subset $\mathscr{M}_s$ of $\mathscr{N}_s$ such that $(\mathscr{N}_s, \mathscr{M}_s)$ is weakly regular in sense of Definition 7.15;

iii) For all $s \in \{1, \ldots, S\}$ the set

$$\mathscr{E}_s := \{c_j : j \in \mathscr{N} \backslash \mathscr{N}_s : \nu_j > \min\{\nu_\kappa : \kappa \in \mathscr{N}, c_\kappa \in P_S \cap \mathscr{E}\}\}$$

has no accumulation point in $P_S$.

See Definition A.2.12. [31, pages 468-469].

**Remark 7.21.** [31, page 469]. We denote the set of vertices of $\mathscr{P}$ by $\widetilde{\mathscr{E}}$ and define

$$\mathscr{M} := \bigcup_{s=1}^{S} \mathscr{M}_s$$

and

$$\widehat{\mathscr{E}} := \{c_j : j \in \mathscr{M}\}.$$

Since the endpoints $P_s$ belong to $\mathscr{M}_s$, we have $\widetilde{\mathscr{E}} \subset \widehat{\mathscr{E}}$.

**Remark 7.22.** Let $D(\lambda)$ be an exponential sum with the representation (7.5) of $b_j$. Assume that the $\nu_j$ do not depend on $j$, i.e., there is a number $\nu \in \mathbb{R}$ such that $\nu_j = \nu$ for all $j \in \mathscr{N}$. Then the exponential sum is weakly regular if $\mathscr{P}$ is a convex polygon. In this case $\widetilde{\mathscr{E}} = \widehat{\mathscr{E}}$. See Remark A.2.13. [31, page 469].

**Theorem 7.23.** *On an unbounded subset $\Omega$ of $\mathbb{C}$ we consider the exponential sum* (7.9)

$$D(\lambda) = \sum_{j \in \mathscr{N}} b_j(\lambda) \exp(c_j \lambda) \quad (\lambda \in \Omega)$$

*fulfilling* (7.4)–(7.8)*, where $b_j(\lambda) = \lambda^{\nu_j}[a_j]$, $\nu_j \in \mathbb{R}$, $a_j \in \mathbb{C}$, $c_j \in \mathbb{C}$. Assume that the exponential sum is weakly regular in the sense of Definition 7.20 and that $a_j \neq 0$ for all $j \in \mathscr{M}$, i.e., $a_j \neq 0$ if $c_j \in \widehat{\mathscr{E}}$. Then the following assertions hold:*

i) *For all* $\lambda \in \mathbb{C}\backslash\{0\}$ *there is a number* $c(\lambda) \in \widetilde{\mathscr{E}}$ *such that for all* $c \in \mathscr{P}$ *the estimate* $\Re((c - c(\lambda))\lambda) \leq 0$ *holds.*

ii) *There are a positive integer* $l$ *and for each* $\delta > 0$ *numbers* $K(\delta) > 0$ *and* $g(\delta) > 0$ *satisfying the following property: for each* $R > K(\delta)$ *there are* $l$ *balls of radius* $\delta$ *such that for all* $\lambda \in \Omega$ *with* $R \leq |\lambda| \leq R + 1$ *outside of these balls the estimate*

$$|\lambda^{-\nu(\lambda)} D(\lambda) \exp\{-c(\lambda)\lambda\}| \geq g(\delta)$$

*holds, where* $\nu(\lambda) := \nu_j$ *if* $c(\lambda) = c_j$*.*

See Theorem A.2.14. [31, page 469].

**Theorem 7.24.** *On an unbounded subset* $\Omega$ *of* $\mathbb{C}$ *we consider the exponential sum* (7.9)

$$D(\lambda) = \sum_{j \in \mathscr{N}} b_j(\lambda) \exp(c_j \lambda) \quad (\lambda \in \Omega)$$

*fulfilling* (7.4)–(7.8)*, where* $b_j(\lambda) = \lambda^{\nu_j}[a_j]$*,* $\nu_j \in \mathbb{R}$*,* $a_j \in \mathbb{C}$*,* $c_j \in \mathbb{C}$*. Assume that the exponential sum is weakly regular in the sense of Definition 7.20 and that* $a_j \neq 0$ *for all* $j \in \mathscr{M}$*, i.e.,* $a_j \neq 0$ *if* $c_j \in \widehat{\mathscr{E}}$*. Then there is an increasing sequence* $(\rho_\nu)_{\nu=1}^{\infty}$ *of positive real numbers with* $\rho_\nu \to \infty$ *as* $\nu \to \infty$ *and a number* $\varepsilon > 0$ *such that for all* $\nu \in \mathbb{N}$ *and all* $\lambda \in \Omega$ *with* $|\lambda| = \rho_\nu$ *there is a number* $c(\lambda) \in \widetilde{\mathscr{E}}$ *such that*

$$\Re((c - c(\lambda))\lambda) \leq 0 \quad \textit{for all} \quad c \in \mathscr{P}$$

*and*

$$|\lambda^{-\nu(\lambda)} D(\lambda) \exp\{-c(\lambda)\lambda\}| \geq \varepsilon,$$

*where* $\nu(\lambda) := \nu_j$ *if* $c(\lambda) = c_j$*.* See Theorem A.2.15. [31, page 472].

## 7.4 Sage codes used for Chapter 5

### 7.4.1 Sage codes used for the computations of $\varphi$ and $\varphi_2$ of Chapter 5

```
# Definition of the variable x
```

```
x = var('x')
nullf(x) = 0

# Definition of the matrix Q0
Q0 = matrix(4,4,1)

# Definition of the matrix epsilon
eps = ones_matrix(4,1)
e={}
for i1 in range (4):
  e[i1] = matrix(4,1)
  e[i1][i1,0] =1

# Definition of the matrix Omega4
Om4 = matrix(4,4,{(0,0):1,(1,1):i,(2,2):-1,(3,3):-i})

# Definition of the matrix V
V = matrix (4,4, lambda i1,i2 : I^(i1*i2))

# Definition of the function g
function('g',x)

# Computations of the functions k1 and k2
k1(x) = -diff(g(x),x)
k2(x) = -g(x)

# Definition of the matrix Q1
Q1 = matrix(4,4,nullf(x))
# use nullf(x) to force this into a matrix of functions;
#any dummy function would do.
#These functions will be overwritten later
```

```
eq55 = g(x)/4*Om4*eps*eps.transpose()*Om4^(-2)*Q0

# Computations of the diagonal entries of Q1 and phi1
for i1 in range(4):
    Q1[i1,i1] = integral((e[i1].transpose()*eq55*e[i1])[0,0],x)
    phi1 = (e[0].transpose()*V*Q1*e[0])[0,0]

# Definition of the matrix Q2
Q2 = matrix(4,4,nullf(x))

# Computations of the non-diagonal entries of Q2
for i1 in range(4):
    for i2 in range(4):
        if i1 != i2:
            Q2[i1,i2] = (diff(Q1[i1,i2],x)\
            -(e[i1].transpose()*eq55*e[i2])[0,0] )/(Om4[i1,i1]-Om4[i2,i2])

eq56a = k2(x)/4*Om4*eps*eps.transpose()*Om4^(-2)*Q1
eq56b = k1(x)/4*Om4*eps*eps.transpose()*Om4^(-3)*Q0
eq56  = eq56a+eq56b

# Computations of the diagonal entries of Q2 and phi2
for i1 in range(4):
    Q2[i1,i1] = integral(-(e[i1].transpose()*eq56*e[i1])[0,0],x)
    phi2 = (e[0].transpose()*V*Q2*e[0])[0,0]
```

### 7.4.2 Sage codes used for the boundary conditions $y''(a) + i\alpha\mu^2 y'(a) = 0$ and $y^{[3]}(a) - i\alpha\mu^2 y(a) = 0$

```
# We denote respectively by al and ep, alha and epsilon
# s=2 denotes the order of the asymptotics
s=2
# Definition of the variables
var('x,mu,al,t,t1,t2,a')
# phi0 is a constant
phi0(x)=1
# Definition of the function phi1
function('phi1',x)
# Definition of the function phi2
function('phi2',x)
# Definition of the list of the functions phis
phi=[phi0(x),phi1(x),phi2(x)]
# Definition of the function g
function('g',x)

# The dictionary of the function eta
eta=[""]
# Definition of the empty list eta
eta={}

# The computation of the solutions of the differential
# equation starts here
for nu in range(1,5):
  eta[nu] = [0]
  for r in range(s+1):
    eta[nu][0]=eta[nu][0]+(mu*i^(nu-1))^(-r)*phi[r]*e^(i^(nu-1)*mu*x)
  for j in range(1,4):
```

```
    eta[nu].append(diff(eta[nu][j-1],x))
# The differential equation ends here

# The dictionary  of the functions deltas
# The empty list Delta
Delta={}
# The empty list delta
delta={}
for nu in range(1,5):
  Delta[nu] = []
  delta[nu] = []
  for j in range(4):
    Delta[nu].append(expand(simplify(eta[nu][j]*e^(-i^(nu-1)*mu*x))*mu^s))
    temp = 0
    for i1 in range(0,s+1):
      temp = temp +  Delta[nu][j].coeff(mu,s+j-i1)*mu^(j-i1)
    delta[nu].append(expand(temp))

#The definition of the boundary conditions
# The empty list of the boundary conditions
gam={}
# The empty list of boundary conditions 1
gam[1] = {}
# The  empty list of boundary conditions 2
gam[2] = {}
# The  empty list of boundary conditions 3
gam[3] = {}
# The  empty list of boundary conditions 4
gam[4] = {}

for k in range(1,5):
```

```
  # The boundary condition y''(a)+i*alpha*lambda*y'(a)=0
  gam[3][k] = delta[k][2].substitute(x=a)+ i*al*mu^2*delta[k][1].substitute(x=a)
  # The boundary condition y^{[3]}(a)-i*alpha*lambda*y(a)=0
  gam[4][k] = delta[k][3].substitute(x=a)-g(a)*delta[k][1].substitute(x=a)\
  - i*al*mu^2*delta[k][0].substitute(x=a)
  for i1 in range(3,5):
    temp = 0
    gam[i1][k]=expand(gam[i1][k]*mu^s);
    m=gam[i1][k].degree(mu)
    for i2 in range(s+1):
      temp = temp + gam[i1][k].coeff(mu,m-i2)*mu^(m-s-i2)
    gam[i1][k] = expand(temp)

# The computation of psi12 starts here
psi0 = expand(gam[3][1]*gam[4][2]-gam[3][2]*gam[4][1])
m1 = psi0.degree(mu);
psi12 = 0
for i1 in range(0,s+1):
  psi12 +=  psi0.coeff(mu,m1-i1)*mu^(m1-i1)
# The computation of psi12 ends here

# The computation of psi42 starts here
psi0 = expand(gam[3][1]*gam[4][4]-gam[3][4]*gam[4][1])
m1 = psi0.degree(mu);
psi42 = 0
for i1 in range(0,s+1):
  psi42 +=  psi0.coeff(mu,m1-i1)*mu^(m1-i1)
# Th e computation of psi42 ends here

# Definition of the lists of the indices p1 and p2 and
# definition of the lists tau0, tau1 and tau2
```

```
p1 = {}
p2 = {}
tau0 = {}
tau1 = {}
tau2 = {}

p1[1]=0
p2[1]=1
tau0[1]=-pi/(4*a)

p1[2]=0
p2[2]=2
tau0[2]=-pi/(2*a)

p1[3]=1
p2[3]=3
tau0[3]=-pi/a

p1[4]=2
p2[4]=3
tau0[4]=-(5*pi)/(4*a)

psi11 = {}
psi41 = {}
psi1 = {}
psi4 = {}
eq0 = {}

for cs in range(1,5):

  for k in range(2,5):
```

```
  # The first boundary condition at the endpoint 0
  gam[1][k]=delta[k][p1[cs]].substitute(x=0)
  # The second boundary condition at the endpoint 0
  gam[2][k]=delta[k][p2[cs]].substitute(x=0)
  if p2[cs] == 3:
    gam[2][k] +=-g(0)*delta[k][1].substitute(x=0)

# The computation of phi11
psi0 = expand((gam[1][3]*gam[2][4]-gam[2][3]*gam[1][4])*mu^(2*s))
m1 = psi0.degree(mu)
psi11[cs] = 0
for i1 in range(0,s+1):
  psi11[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

# The computation of phi41
psi0 = expand((gam[1][2]*gam[2][3]-gam[2][2]*gam[1][3])*mu^(2*s))
m1 = psi0.degree(mu)
psi41[cs] = 0
for i1 in range(0,s+1):
  psi41[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

# The computation of psi1
psi0 = expand(psi11[cs]*psi12)
m1 = psi0.degree(mu)
psi1[cs] = 0
for i1 in range(s+1):
  psi1[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

# The computation of psi4
psi0 = expand(psi41[cs]*psi42)
m1 = psi0.degree(mu)
```

```
psi4[cs] = 0
for i1 in range(s+1):
  psi4[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

m1 = psi1[cs].degree(mu)

t0 = tau0[cs]
# The computation of the Taylor expansion of e^{-2*i*mu_k*a}
f1 = taylor(exp(-2*i*t0*a)*exp(-2*i*a*(t1*t+t2*t^2)),t,0,2);

# The computation of the Taylor expansion of  1/mu_k
minv = taylor((1+a*(t0+t1*t)*t/pi)^(-1),t,0,1)
muinv = minv*a*t/pi

# The computation of mu_k^(-m1)*psi1(mu_k)
psi1b = (expand(psi1[cs]/mu^m1)).substitute(mu=1/muinv)
psi1a =0
for i1 in range(s+1):
  psi1a += (diff(psi1b,t,i1)).substitute(t=0)*t^i1/factorial(i1)


# The computation of mu_k^(-m1)*psi4(mu_k)
psi4b = (expand(psi4[cs]/mu^m1)).substitute(mu=1/muinv)
psi4a =0
for i1 in range(s+1):
  psi4a += (diff(psi4b,t,i1)).substitute(t=0)*t^i1/factorial(i1)

# The computation of D1(mu_k)=0
D1 = psi1a+psi4a*f1
eq0[cs] = D1.substitute(t=0)
eq1 = (diff(D1,t)).substitute(t=0)
```

```
  eq2 = (diff(D1,t,2)).substitute(t=0)

  # The computations of tau1 and tau2
  tau1[cs] = expand(solve(eq1,t1)[0].right())
  temp = eq2.subs(t1=tau1[cs])
  tau2[cs] = expand(-temp.coefficient(t2,0)/temp.coefficient(t2,1))
```

### 7.4.3 Sage codes used for the boundary conditions $y''(a) + i\alpha\mu^2 y'(a) = 0$ and $y(a) + i\alpha\mu^2 y^{[3]}(a) = 0$

```
# We denote respectively by al and ep, alpha and epsilon
# s=2  denotes the order of the asymptotics
s=2
# Definition of the variables
var('x,mu,al,t,t1,t2,a')
# phi0 is a constant
phi0(x)=1
# Definition of the function phi1
function('phi1',x)
# Definition of the function phi2
function('phi2',x)
# Definition of the list of the functions phi
phi=[phi0(x),phi1(x),phi2(x)]
# Definition of the function g
function('g',x)

# The dictionary of the function eta
eta=[""]

# Definition of the empty list eta
```

```
eta={}

# The computation of the solutions of the differential equation
# starts here
for nu in range(1,5):
  eta[nu] = [0]
  for r in range(s+1):
    eta[nu][0]=eta[nu][0]+(mu*i^(nu-1))^(-r)*phi[r]*e^(i^(nu-1)*mu*x)
  for j in range(1,4):
    eta[nu].append(diff(eta[nu][j-1],x))

# The dictionary of the functions deltas
# The empty list Delta
Delta={}
# The empty list delta
delta={}
for nu in range(1,5):
  Delta[nu] = []
  delta[nu] = []
  for j in range(4):
    Delta[nu].append(expand(simplify(eta[nu][j]*e^(-i^(nu-1)*mu*x))*mu^s))
    temp = 0
    for i1 in range(0,s+1):
      temp = temp +  Delta[nu][j].coeff(mu,s+j-i1)*mu^(j-i1)
    delta[nu].append(expand(temp))

# The definition of the boundary conditions
# The empty list of the boundary conditions
gam={}
# The empty list of the boundary condition 1
gam[1] = {}
```

```
# The empty list of the boundary condition 2
gam[2] = {}
# The empty list of the boundary condition 3
gam[3] = {}
# The empty list of the boundary condition 4
gam[4] = {}

for k in range(1,5):
  # The boundary condition y''(a)+i*alpha*mu^2*y'(a)=0
  gam[3][k] = delta[k][2].substitute(x=a)+ i*al*mu^2*delta[k][1].substitute(x=a)
  # The boundary condition y(a)+i*alpha*mu^2*y^{[3]}(a)=0
  gam[4][k] = delta[k][0].substitute(x=a)+i*al*mu^2*(delta[k][3].substitute(x=a)\
  -g(a)*delta[k][1].substitute(x=a))

  for i1 in range(3,5):
    temp = 0
    gam[i1][k]=expand(gam[i1][k]*mu^s);
    m=gam[i1][k].degree(mu)
    for i2 in range(s+1):
      temp = temp + gam[i1][k].coeff(mu,m-i2)*mu^(m-s-i2)
    gam[i1][k] = expand(temp)

# The computation of psi12 starts here
psi0 = expand(gam[3][1]*gam[4][2]-gam[3][2]*gam[4][1])
m1 = psi0.degree(mu);
psi12 = 0
for i1 in range(0,s+1):
  psi12 +=  psi0.coeff(mu,m1-i1)*mu^(m1-i1)
# The computation of psi12 ends here

# The computation of psi42 starts here
```

```
psi0 = expand(gam[3][1]*gam[4][4]-gam[3][4]*gam[4][1])
m1 = psi0.degree(mu);
psi42 = 0
for i1 in range(0,s+1):
  psi42 +=  psi0.coeff(mu,m1-i1)*mu^(m1-i1)
# The computation of psi42 ends here

# Definition of the lists of the indices p1 and p2 and
# definition of the lists tau0, tau1 and tau2
p1 = {}
p2 = {}
tau0 = {}
tau1 = {}
tau2 = {}

p1[1]=0
p2[1]=1
tau0[1]=-(5*pi)/(4*a)

p1[2]=0
p2[2]=2
tau0[2]=-(3*pi)/(2*a)

p1[3]=1
p2[3]=3
tau0[3]=-(2*pi)/a

p1[4]=2
p2[4]=3
tau0[4]=-(9*pi)/(4*a)
```

```
psi11 = {}

psi41 = {}

psi1 = {}

psi4 = {}

eq0 = {}


for cs in range(1,5):


  for k in range(2,5):

    # The first boundary condition at the endpoint 0

    gam[1][k]=delta[k][p1[cs]].substitute(x=0)

    # The second boundary condition at the endpoint 0

    gam[2][k]=delta[k][p2[cs]].substitute(x=0)

    if p2[cs] == 3:

      gam[2][k] +=-g(0)*delta[k][1].substitute(x=0)


# The computation of phi11

psi0 = expand((gam[1][3]*gam[2][4]-gam[2][3]*gam[1][4])*mu^(2*s))

m1 = psi0.degree(mu)

psi11[cs] = 0

for i1 in range(0,s+1):

    psi11[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)


# The computation of phi41

psi0 = expand((gam[1][2]*gam[2][3]-gam[2][2]*gam[1][3])*mu^(2*s))

m1 = psi0.degree(mu)

psi41[cs] = 0

for i1 in range(0,s+1):

    psi41[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)


# The computation of phi1
```

```
psi0 = expand(psi11[cs]*psi12)
m1 = psi0.degree(mu)
psi1[cs] = 0
for i1 in range(s+1):
    psi1[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

# The computation of phi4
psi0 = expand(psi41[cs]*psi42)
m1 = psi0.degree(mu)
psi4[cs] = 0
for i1 in range(s+1):
    psi4[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

m1 = psi1[cs].degree(mu)

t0 = tau0[cs]
# The computation of the Taylor expansion of e^{-2*i*mu_k*a}
f1 = taylor(exp(-2*i*t0*a)*exp(-2*i*a*(t1*t+t2*t^2)),t,0,2);

# The computation of the Taylor expansion of 1/mu_k
minv = taylor((1+a*(t0+t1*t)*t/pi)^(-1),t,0,1)
muinv = minv*a*t/pi

# The computation of mu_k^(-m1)*psi1(mu_k)
psi1b = (expand(psi1[cs]/mu^m1)).substitute(mu=1/muinv)
psi1a =0
for i1 in range(s+1):
    psi1a += (diff(psi1b,t,i1)).substitute(t=0)*t^i1/factorial(i1)

# The computation of mu_k^(-m1)*psi4(mu_k)
psi4b = (expand(psi4[cs]/mu^m1)).substitute(mu=1/muinv)
```

```
psi4a =0
for i1 in range(s+1):
    psi4a += (diff(psi4b,t,i1)).substitute(t=0)*t^i1/factorial(i1)

# The computation of D1(mu_k)=0
D1 = psi1a+psi4a*f1
eq0[cs] = D1.substitute(t=0)
eq1 = (diff(D1,t)).substitute(t=0)
eq2 = (diff(D1,t,2)).substitute(t=0)

# The computations of tau1 and tau2
tau1[cs] = expand(solve(eq1,t1)[0].right())
temp = eq2.subs(t1=tau1[cs])
tau2[cs] = expand(-temp.coefficient(t2,0)/temp.coefficient(t2,1))
```

### 7.4.4 Sage codes used for the boundary conditions $y'(a) - i\alpha\mu^2 y''(a) = 0$ and $y^{[3]}(a) - i\alpha\mu^2 y(a) = 0$

```
# We denote respectively by al and ep, alpha and epsilon
# s=2 denotes the order of the asymptotics
s=2
# Definition of the variables
var('x,mu,al,t,t1,t2,a')
# phi0 is a constant
phi0(x)=1
# Definition of the function phi1
function('phi1',x)
# Definition of the function phi2
function('phi2',x)
# Definition of the list of the functions phis
```

```
phi=[phi0(x),phi1(x),phi2(x)]
# Definition of the function g
function('g',x)

# The dictionary of the function eta
eta=[""]
# Definition of the empty list eta
eta={}

# The computation of the solutions of the differential
# equation starts here
for nu in range(1,5):
  eta[nu] = [0]
  for r in range(s+1):
    eta[nu][0]=eta[nu][0]+(mu*i^(nu-1))^(-r)*phi[r]*e^(i^(nu-1)*mu*x)
  for j in range(1,4):
    eta[nu].append(diff(eta[nu][j-1],x))
# The differential equation ends here

# The dictionary of the functions deltas
# The empty list Delta
Delta={}
# The empty list delta
delta={}
for nu in range(1,5):
  Delta[nu] = []
  delta[nu] = []
  for j in range(4):
    Delta[nu].append(expand(simplify(eta[nu][j]*e^(-i^(nu-1)*mu*x))*mu^s))
    temp = 0
    for i1 in range(0,s+1):
```

```
      temp = temp +  Delta[nu][j].coeff(mu,s+j-i1)*mu^(j-i1)
    delta[nu].append(expand(temp))

# The definition of the boundary conditions
# The empty list of the boundary conditions
gam={}
# The empty list of the boundary condition 1
gam[1] = {}
# The empty list of the boundary condition 2
gam[2] = {}
# The empty list of the boundary condition 3
gam[3] = {}
# The empty list of the boundary condition 4
gam[4] = {}

for k in range(1,5):
  # The boundary condition y'(a)-i*alpha*mu^2*y''(a)=0
  gam[3][k] = delta[k][1].substitute(x=a)- i*al*mu^2*delta[k][2].substitute(x=a)
  # The boundary condition y^{[3]}(a)-i*alpha*mu^2*y(a)=0
  gam[4][k] = delta[k][3].substitute(x=a)-g(a)*delta[k][1].substitute(x=a)\
  - i*al*mu^2*delta[k][0].substitute(x=a)
  for i1 in range(3,5):
    temp = 0
    gam[i1][k]=expand(gam[i1][k]*mu^s);
    m=gam[i1][k].degree(mu)
    for i2 in range(s+1):
      temp = temp + gam[i1][k].coeff(mu,m-i2)*mu^(m-s-i2)
    gam[i1][k] = expand(temp)

# The computation of psi12 starts here
psi0 = expand(gam[3][1]*gam[4][2]-gam[3][2]*gam[4][1])
```

```
m1 = psi0.degree(mu);
psi12 = 0
for i1 in range(0,s+1):
  psi12 +=  psi0.coeff(mu,m1-i1)*mu^(m1-i1)
# The computation of psi12 ends here

# The computation of psi42 starts here
psi0 = expand(gam[3][1]*gam[4][4]-gam[3][4]*gam[4][1])
m1 = psi0.degree(mu);
psi42 = 0
for i1 in range(0,s+1):
  psi42 +=  psi0.coeff(mu,m1-i1)*mu^(m1-i1)
# The computation of psi42 ends here

# Definition of the lists of the indices p1 and p2 and
# definition of the lists tau1 and tau2
p1 = {}
p2 = {}
tau0 = {}
tau1 = {}
tau2 = {}

p1[1]=0
p2[1]=1
tau0[1]=-pi/(2*a)

p1[2]=0
p2[2]=2
tau0[2]=-(3*pi)/(4*a)

p1[3]=1
```

```
p2[3]=3
tau0[3]=-(5*pi)/(4*a)

p1[4]=2
p2[4]=3
tau0[4]=-(5*pi)/(2*a)

psi11 = {}
psi41 = {}
psi1 = {}
psi4 = {}
eq0 = {}

for cs in range(1,5):

  for k in range(2,5):
    # The first boundary condition at the endpoint 0
    gam[1][k]=delta[k][p1[cs]].substitute(x=0)
    # The second boundary condition at the endpoint 0
    gam[2][k]=delta[k][p2[cs]].substitute(x=0)
    if p2[cs] == 3:
      gam[2][k] +=-g(0)*delta[k][1].substitute(x=0)

# The computation of psi11
psi0 = expand((gam[1][3]*gam[2][4]-gam[2][3]*gam[1][4])*mu^(2*s))
m1 = psi0.degree(mu)
psi11[cs] = 0
for i1 in range(0,s+1):
    psi11[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

# The computation of psi41
```

```
psi0 = expand((gam[1][2]*gam[2][3]-gam[2][2]*gam[1][3])*mu^(2*s))
m1 = psi0.degree(mu)
psi41[cs] = 0
for i1 in range(0,s+1):
    psi41[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

# The computation of psi1
psi0 = expand(psi11[cs]*psi12)
m1 = psi0.degree(mu)
psi1[cs] = 0
for i1 in range(s+1):
    psi1[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

# The computation of psi4
psi0 = expand(psi41[cs]*psi42)
m1 = psi0.degree(mu)
psi4[cs] = 0
for i1 in range(s+1):
    psi4[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

m1 = psi1[cs].degree(mu)

t0 = tau0[cs]
# The Taylor expansion of e^{-2*i*mu_k*a}
f1 = taylor(exp(-2*i*t0*a)*exp(-2*i*a*(t1*t+t2*t^2)),t,0,2);

# The Taylor expansion of 1/mu_k
minv = taylor((1+a*(t0+t1*t)*t/pi)^(-1),t,0,1)
muinv = minv*a*t/pi

# The computation of mu_k^(-m1)*psi1(mu_k)
```

```
psi1b = (expand(psi1[cs]/mu^m1)).substitute(mu=1/muinv)
psi1a =0
for i1 in range(s+1):
    psi1a += (diff(psi1b,t,i1)).substitute(t=0)*t^i1/factorial(i1)

# The computation of mu_k^(-m1)*psi4(mu_k)
psi4b = (expand(psi4[cs]/mu^m1)).substitute(mu=1/muinv)
psi4a =0
for i1 in range(s+1):
    psi4a += (diff(psi4b,t,i1)).substitute(t=0)*t^i1/factorial(i1)

# The computation of D1(mu_k)=0
D1 = psi1a+psi4a*f1
eq0[cs] = D1.substitute(t=0)
eq1 = (diff(D1,t)).substitute(t=0)
eq2 = (diff(D1,t,2)).substitute(t=0)

# The computation of tau1 and tau2
tau1[cs] = expand(solve(eq1,t1)[0].right())
temp = eq2.subs(t1=tau1[cs])
tau2[cs] = expand(-temp.coefficient(t2,0)/temp.coefficient(t2,1))
```

### 7.4.5 Sage codes used for the boundary conditions $y'(a) - i\alpha\mu^2 y''(a) = 0 = 0$ and $y(a) + i\alpha\mu^2 y^{[3]}(a) = 0$

```
#we respectively denote by al  and ep, alpha and epsilon
# s=2 denotes the order of the asymptotics
s=2
# Definition of the variables
var('x,mu,al,t,t1,t2,a')
```

```
# phi0 is a constant
phi0(x)=1
# Definition of the function phi1
function('phi1',x)
# Definition of the function phi2
function('phi2',x)
# Definition of the list of functions phis
phi=[phi0(x),phi1(x),phi2(x)]
# Definition of the function g
function('g',x)

# The definition of the function eta
eta=[""]
# Definition of the empty list eta
eta={}

# The computation of the solutions of the differential
# equation starts here
for nu in range(1,5):
  eta[nu] = [0]
  for r in range(s+1):
    eta[nu][0]=eta[nu][0]+(mu*i^(nu-1))^(-r)*phi[r]*e^(i^(nu-1)*mu*x)
  for j in range(1,4):
    eta[nu].append(diff(eta[nu][j-1],x))
# The differential equation ends here

# The dictionary of the functions deltas
# The empty list Delta
Delta={}
# The empty list delta
delta={}
```

```
for nu in range(1,5):
  Delta[nu] = []
  delta[nu] = []
  for j in range(4):
    Delta[nu].append(expand(simplify(eta[nu][j]*e^(-i^(nu-1)*mu*x))*mu^s))
    temp = 0
    for i1 in range(0,s+1):
      temp = temp +  Delta[nu][j].coeff(mu,s+j-i1)*mu^(j-i1)
    delta[nu].append(expand(temp))

# The definition of the boundary conditions
# The empty list of the boundary conditions
gam={}
# The empty list of the boundary condition 1
gam[1] = {}
# The empty list of the boundary condition 2
gam[2] = {}
# The empty list of the boundary condition 3
gam[3] = {}
# The empty list of the boundary condition 4
gam[4] = {}

for k in range(1,5):
  # The boundary condition y'(a)-i*alpha*mu^2*y''(a)=0
  gam[3][k] = delta[k][1].substitute(x=a)- i*al*mu^2*delta[k][2].substitute(x=a)
  # The boundary condition y(a)+i*alpha*mu^2*y^{[3]}(a)=0
  gam[4][k] = delta[k][0].substitute(x=a)+i*al*mu^2*(delta[k][3].substitute(x=a)\
  -g(a)*delta[k][1].substitute(x=a))
  for i1 in range(3,5):
    temp = 0
    gam[i1][k]=expand(gam[i1][k]*mu^s);
```

```
    m=gam[i1][k].degree(mu)
    for i2 in range(s+1):
      temp = temp + gam[i1][k].coeff(mu,m-i2)*mu^(m-s-i2)
    gam[i1][k] = expand(temp)

# The computation of psi12 starts here
psi0 = expand(gam[3][1]*gam[4][2]-gam[3][2]*gam[4][1])
m1 = psi0.degree(mu);
psi12 = 0
for i1 in range(0,s+1):
  psi12 +=  psi0.coeff(mu,m1-i1)*mu^(m1-i1)
# The computation of psi12 ends here

# The computation of psi42 starts here
psi0 = expand(gam[3][1]*gam[4][4]-gam[3][4]*gam[4][1])
m1 = psi0.degree(mu);
psi42 = 0
for i1 in range(0,s+1):
  psi42 +=  psi0.coeff(mu,m1-i1)*mu^(m1-i1)
# The computation of psi42 ends here

# Definition of the list of the indices p1 and p2
# and definition of the lists tau0, tau1 and tau2
p1 = {}
p2 = {}
tau0 = {}
tau1 = {}
tau2 = {}

p1[1]=0
p2[1]=1
```

```
tau0[1]=-(3*pi)/(2*a)

p1[2]=0
p2[2]=2
tau0[2]=-(7*pi)/(4*a)

p1[3]=1
p2[3]=3
tau0[3]=-(9*pi)/(4*a)

p1[4]=2
p2[4]=3
tau0[4]=-(7*pi)/(2*a)

psi11 = {}
psi41 = {}
psi1 = {}
psi4 = {}
eq0 = {}

for cs in range(1,5):

  for k in range(2,5):
    # The first boundary  condition at the endpoint 0
    gam[1][k]=delta[k][p1[cs]].substitute(x=0)
    # The second boundary  condition at the endpoint 0
    gam[2][k]=delta[k][p2[cs]].substitute(x=0)
    if p2[cs] == 3:
      gam[2][k] +=-g(0)*delta[k][1].substitute(x=0)

# The computation of phi11
```

```
psi0 = expand((gam[1][3]*gam[2][4]-gam[2][3]*gam[1][4])*mu^(2*s))
m1 = psi0.degree(mu)
psi11[cs] = 0
for i1 in range(0,s+1):
    psi11[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

# The computation of phi41
psi0 = expand((gam[1][2]*gam[2][3]-gam[2][2]*gam[1][3])*mu^(2*s))
m1 = psi0.degree(mu)
psi41[cs] = 0
for i1 in range(0,s+1):
    psi41[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

# The computation of phi1
psi0 = expand(psi11[cs]*psi12)
m1 = psi0.degree(mu)
psi1[cs] = 0
for i1 in range(s+1):
    psi1[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

# The computation of phi4
psi0 = expand(psi41[cs]*psi42)
m1=psi0.degree(mu)
psi4[cs] = 0
for i1 in range(s+1):
    psi4[cs] += psi0.coefficient(mu,m1-i1)*mu^(m1-i1)

m1 = psi1[cs].degree(mu)

t0 = tau0[cs]
# The computation of the Taylor expansion of e^{-2*i*mu_k*a}
```

```
f1 = taylor(exp(-2*i*t0*a)*exp(-2*i*a*(t1*t+t2*t^2)),t,0,2);

# The computation of the Taylor expansion of 1/mu_k
minv = taylor((1+a*(t0+t1*t)*t/pi)^(-1),t,0,1)
muinv = minv*a*t/pi

# The computation of mu_k^(-m1)*psi1(mu_k)
psi1b = (expand(psi1[cs]/mu^m1)).substitute(mu=1/muinv)
psi1a =0
for i1 in range(s+1):
    psi1a += (diff(psi1b,t,i1)).substitute(t=0)*t^i1/factorial(i1)

# The computation of mu_k^(-m1)*psi4(mu_k)
psi4b = (expand(psi4[cs]/mu^m1)).substitute(mu=1/muinv)
psi4a =0
for i1 in range(s+1):
    psi4a += (diff(psi4b,t,i1)).substitute(t=0)*t^i1/factorial(i1)

# The computation of D1(mu_k)
D1 = psi1a+psi4a*f1
eq0[cs] = D1.substitute(t=0)
eq1 = (diff(D1,t)).substitute(t=0)
eq2 = (diff(D1,t,2)).substitute(t=0)

# The computation of tau1 and tau2
tau1[cs] = expand(solve(eq1,t1)[0].right())
temp = eq2.subs(t1=tau1[cs])
tau2[cs] = expand(-temp.coefficient(t2,0)/temp.coefficient(t2,1))
```

## 7.5 Sage codes used for Chapter 6

### 7.5.1 Sage codes used for the computation of the functions $\varphi_1$ and $\varphi_2$

```
# Definition of the variable x
x=var('x')
nullf(x)=0
#Definition of the matrix Q0
Q0=matrix(6,6,1)
# Definition of the matrix epsilon
eps=ones_matrix(6,1)
e={}
for i1 in range (6):
    e[i1]=matrix(6,1)
    e[i1][i1,0]=1

# Definition of the 6 by 6 matrix Omega6
Om6=matrix(6,6,{(0,0):1,(1,1):(1+sqrt(3)*i)/2,(2,2):-(1-sqrt(3)*i)/2,(3,3):\
-1,(4,4):-(1+sqrt(3)*i)/2,(5,5):(1-sqrt(3)*i)/2})

# Definition of the 6 by 6 matrix V
V=matrix(6,6, lambda i1,i2  : ((1+sqrt(3)*i)/2)^(i1*i2))

# Definition of the function g
function('g',x)

# Computations of the functions k3 and k4 of the differential equation
k3(x)=2*diff(g(x),x)
k4(x)=g(x)
# Definition of the matrix Q1
Q1=matrix(6,6,nullf(x))# use nullf(x) to force this into matrix of functions;
```

```
eq77=g(x)/6*Om6*eps*eps.transpose()*Om6^(-2)*Q0

# Computations of the diagonal elements of Q2 and  phi1
for i1 in range(6):
    Q1[i1,i1]=integral((-e[i1].transpose()*eq77*e[i1])[0,0],x)
    phi1=(e[0].transpose()*V*Q1*e[0])[0,0]

# Definition of the matrix Q2
Q2=matrix(6,6,nullf(x))

# Computation of the non-diagonal entries of Q2
for i1 in range(6):
    for i2 in range(6):
        if i1!=i2:
            Q2[i1,i2]=(diff(Q1[i1,i2],x)+\
            (e[i1].transpose()*eq77*e[i2])[0,0])/(Om6[i1,i1]-Om6[i2,i2])


eq78a=k4(x)/6*Om6*eps*eps.transpose()*Om6^(-2)*Q1
eq78b=k3(x)/6*Om6*eps*eps.transpose()*Om6^(-3)*Q0
eq78=eq78a+eq78b

# Computations of the diagonal entries of Q2 and phi2
for i1 in range(6):
    Q2[i1,i1]=integral(-(e[i1].transpose()*eq78*e[i1])[0,0],x)
K=V*Q2
phi2=(e[0].transpose()*V*Q2*e[0])[0,0]
```

### 7.5.2 Sage codes used for the computation of the first four terms of the eigenvalues asymptotics of Chapter 6

```
# We denote by respectively by al and ep, alpha and epsilon
#asymptotic of order 2
s=2
#Definition of the variables
var('x,mu,al,ep,t,u,t1,t2,a')
#phi0 is a constant
phi0(x)=1
#Definition  of the function phi1
function('phi1',x)
#Definition of the function phi2
function('phi2',x)
#Definition of the list of functions phis
phi=[phi0(x),phi1(x),phi2(x)]


# The dictionary of the functions eta
#Definition of the empty list eta
eta={}
#The building of the solution of the differential equation starts here
for nu in range(1,7):
    eta[nu]=[0];
    for r in range(s+1):
        eta[nu][0]=eta[nu][0]+(mu*exp(pi*i*(nu-1)/3))^(-r)\
        *phi[r]*exp(exp(i*pi*(nu-1)/3)*mu*x)
    for j in range(1,6):
            eta[nu].append(diff(eta[nu][j-1],x))
#end of the differential equation
```

```
# The dictionary of the functions deltas
#The empty list Delta
Delta={}
# The empty list delta
delta={}
for nu in range(1,7):
    Delta[nu]=[]
    delta[nu]=[]
    for j in range(6):
        Delta[nu].append(expand(simplify(eta[nu][j]\
        *exp(-exp(i*pi*(nu-1)/3)*mu*x))*mu^s))
        temp=0
        for i1 in range(0,s+1):
            temp=temp + Delta[nu][j].coeff(mu,s+j-i1)*mu^(j-i1)
        delta[nu].append(expand(temp))

# The definition of the boundary conditions
#The empty list of the boundary conditions
cond={}
#The boundary condition 1
cond[1]={}
#The boundary condition 2
cond[2]={}
#The boundary condition 3
cond[3]={}
#The boundary condition 4
cond[4]={}
#The boundary condition 5
cond[5]={}
#The boundary condition 6
cond[6]={}
```

```
for k in range(1,7):
    #Boundary term y(0)
    cond[1][k]=delta[k][0].substitute(x=0)
    #Boundary term y'(0)
    cond[2][k]=delta[k][1].substitute(x=0)
    #Boundary term y''(0)
    cond[3][k]=delta[k][2].substitute(x=0)
    #Boundary term y(a)
    cond[4][k]=delta[k][0].substitute(x=a)
    #Boundary term y''(a)
    cond[5][k]=delta[k][2].substitute(x=a)
    #Boundary term y^{(4)}(a)-i*alpha*lambda*y'(a)
    cond[6][k]=delta[k][4].substitute(x=a)+ep*al*mu^3*delta[k][1].substitute(x=a)

    for i1 in range(1,7):
        temp=0
        cond[i1][k]=expand(cond[i1][k]*mu^s);
        m=cond[i1][k].degree(mu)
        for i2 in range(s+1):
            temp=temp+cond[i1][k].coeff(mu,m-i2)*mu^(m-s-i2)
        cond[i1][k]=expand(temp)

# The computation of psi1 starts here

psi0=expand(cond[4][1]*(cond[5][2]*cond[6][6]-cond[6][2]*cond[5][6])\
-cond[4][2]*(cond[5][1]*cond[6][6]-cond[6][1]*cond[5][6])\
+cond[4][6]*(cond[5][1]*cond[6][2]-cond[6][1]*cond[5][2]))
m1=psi0.degree(mu);
psi11=0
for i1 in range(0,s+1):
    psi11+=psi0.coeff(mu,m1-i1)*mu^(m1-i1)
```

```
psi0=expand(cond[1][3]*(cond[2][4]*cond[3][5]-cond[3][4]*cond[2][5])\
-cond[1][4]*(cond[2][3]*cond[3][5]-cond[3][3]*cond[2][5])\
+cond[1][5]*(cond[2][3]*cond[3][4]-cond[3][3]*cond[2][4]))*mu^(2*s)
m1=psi0.degree(mu);
psi12=0
for i1 in range(0,s+1):
    psi12+=psi0.coeff(mu,m1-i1-2*s)*mu^(m1-i1-2*s)

psi10=expand(psi11*psi12)
m1=psi10.degree(mu)
psi1=0
for i1 in range(s+1):
    psi1+=psi10.coeff(mu,m1-i1)*mu^(m1-i1)
# The computation of psi1 finishes here

# The computation of psi2 starts here

psi0=expand(cond[4][1]*(cond[5][2]*cond[6][3]-cond[6][2]*cond[5][3])\
-cond[4][2]*(cond[5][1]*cond[6][3]-cond[6][1]*cond[5][3])\
+cond[4][3]*(cond[5][1]*cond[6][2]-cond[6][1]*cond[5][2]))
m1=psi0.degree(mu);
psi21=0
for i1 in range(0,s+1):
    psi21+=psi0.coeff(mu,m1-i1)*mu^(m1-i1)


psi0=expand(cond[1][4]*(cond[2][5]*cond[3][6]-cond[3][5]*cond[2][6])\
-cond[1][5]*(cond[2][4]*cond[3][6]-cond[3][4]*cond[2][6])\
+cond[1][6]*(cond[2][4]*cond[3][5]-cond[3][4]*cond[2][5]))*mu^(2*s)
m1=psi0.degree(mu)
```

```
psi22=0
for i1 in range(0,s+1):
  psi22+=psi0.coeff(mu,m1-i1-2*s)*mu^(m1-i1-2*s)

psi20=expand(psi21*psi22)
m2=psi20.degree(mu)
psi2=0
for i1 in range(s+1):
    psi2+=psi20.coeff(mu,m2-i1)*mu^(m2-i1)
# The computation of psi2 finishes here

m1=psi1.degree(mu)
m2=psi2.degree(mu)

#The computation of tau1 and tau2 for alpha>3 starts here,
#we denote respectively by tau1g and tau2g, tau1 and tau2
#for alpha>3

# The Taylor expansion of exp((-1+i*sqrt(3))*mu_k*a
#for alpha>3 is done here
tau0g=-(1+i*sqrt(3))/(4*a)*ln(abs((ep*al-3)/(ep*al+3)))
t0g=tau0g
f1g=taylor(exp((-1+i*sqrt(3))*t0g*a)\
*exp((-1+i*sqrt(3))*(t1*t+t2*t^2)*a),t,0,2)


# The Taylor expansion of 1/mu_k is done here for alpha >3
f2g=(1-1/(2*pi)*i*t*(ln(abs((ep*al-3)/(ep*al+3)))))^(-1)
minvg = f2g.taylor(t,0,2)
muinvg=(sqrt(3)+i)*a*t/(2*pi)*minvg
```

```
#The computation of mu_k^(-m1)*psi1(mu_k)
#is done here for alpha>3
psi1bg=(expand(psi1/mu^m1)).substitute(mu=1/muinvg)
m0bg=psi1bg.degree(mu)
m1bg=psi1bg.degree(t)
psi1ag=0
for i1 in range(s+1):
    psi1ag+=(diff(psi1bg,t,i1)).substitute(t=0)*t^i1/factorial(i1)
m1a=psi1.degree(t)

#The computation of mu_k^(-m1)*psi2(mu_k) is done here for alpha>3
psi2bg=(expand(psi2/mu^m1)).substitute(mu=1/muinvg)
m2bg=psi2bg.degree(t)
psi2ag=0
for i1 in range(s+1):
    psi2ag+=(diff(psi2bg,t,i1)).substitute(t=0)*t^i1/factorial(i1)
m2ag=psi2ag.degree(t)

#The computation of
#mu_k^(-m1)*psi1(mu_k)+mu_k^(-m1)*psi2(mu_k)*e^((-1+i*sqrt(3))*mu_k*a)
#is done here
D1g=psi1ag+psi2ag*f1g



eq0g=D1g.substitute(t=0)
eq1g=(diff(D1g,t)).substitute(t=0)
eq2g=(diff(D1g,t,2)).substitute(t=0)

# The computations of tau1 and tau2 for alpha>3 are done here
tau1g=expand(solve(eq1g,t1)[0].right())
temp=eq2g.subs(t1=tau1g)
```

```
tau2g=expand(-temp.coefficient(t2,0)/temp.coefficient(t2,1))

#The computation of tau1 and tau2 for alpha<3 starts here,
#we denote respectively by tau1l and tau2l, tau1 and tau2 for alpha<3

# The Taylor expansion of exp((-1+i*sqrt(3))*mu_k*a for alpha<3 is done here
tau0l=-(1+i*sqrt(3))/(4*a)*(ln(abs((ep*al-3)/(ep*al+3)))+i*pi)
t0l=tau0l
f1l=taylor(exp((-1+i*sqrt(3))*t0l*a)\
*exp((-1+i*sqrt(3))*(t1*t+t2*t^2)*a),t,0,2)

# The Taylor expansion of 1/mu_k is done here for alpha <3
f2l=(1-1/(2*pi)*i*t*(ln(abs((ep*al-3)/(ep*al+3)))+i*pi))^(-1)
minvl = f2l.taylor(t,0,2)
muinvl=(sqrt(3)+i)*a*t/(2*pi)*minvl

#The computation of mu_k^(-m1)*psi1(mu_k) is done here for alpha<3
psi1bl=(expand(psi1/mu^m1)).substitute(mu=1/muinvl)
m0bl=psi1bl.degree(mu)
m1bl=psi1bl.degree(t)
psi1al=0
for i1 in range(s+1):
    psi1al+=(diff(psi1bl,t,i1)).substitute(t=0)*t^i1/factorial(i1)
m1al=psi1al.degree(t)


#The computation of mu_k^(-m1)*psi2(mu_k) is done here for alpha<3
psi2bl=(expand(psi2/mu^m1)).substitute(mu=1/muinvl)
m2bl=psi2bl.degree(t)
psi2al=0
for i1 in range(s+1):
```

```
    psi2al+=(diff(psi2bl,t,i1)).substitute(t=0)*t^i1/factorial(i1)
m2al=psi2al.degree(t)

#The computation of
#mu_k^(-m1)*psi1(mu_k)+mu_k^(-m1)*psi2(mu_k)*e^((-1+i*sqrt(3))*mu_k*a)
#is done here
D1l=psi1al+psi2al*f1l



eq0l=D1l.substitute(t=0)
eq1l=(diff(D1l,t)).substitute(t=0)
eq2l=(diff(D1l,t,2)).substitute(t=0)

# The computations of tau1 and tau2 for alpha<3 are done here
tau1l=expand(solve(eq1l,t1)[0].right())
temp=eq2l.subs(t1=tau1l)
tau2l=expand(-temp.coefficient(t2,0)/temp.coefficient(t2,1))
```

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